			$C = i\text{-}\bar{i}(010)$. Meas.	Calc.	$r^3 = \bar{2}43$. Meas.	Calc.	$i^2=201$. Meas.	Calc.
i^2	$2\text{-}\bar{i}$	021	17° 37′	17° 38½	23° 58′	23° 49′		
$+x^1$	$\frac{12}{7}\text{-}6$	2·12·7	22 9	21 53	31 4	31 12	8° 21′	8° 37′
$+x^2$	$\frac{26}{9}\text{-}13$	2·26·9	12 55	12 58	29 32	29 27	6 36	6 35
$+x^3$	$\frac{26}{7}\text{-}13$	2·26·7	10 34	10 30	30 37	30 42	8 50	8 54
$+x^4$	$\frac{34}{7}\text{-}\frac{17}{2}$	4·34·7	9 42	9 39	33 41	33 41	12 0	11 53
$-x^6$	$\frac{12}{7}\text{-}24$	$\bar{1}$·24·14	20 21	20 28	21 11	21 6	3 25	3 24
$-x^5$	$\frac{13}{12}\text{-}\frac{13}{5}$	$\bar{5}$·13·12	34 32	34 27	7 32	7 15	21 15	21 8
$-x^7$	$\frac{13}{7}\text{-}13$	$\bar{1}$·13·7	18 50	19 17	19 40	19 47	4 5	4 3
$-x^8$	$\frac{24}{13}\text{-}4$	$\bar{6}$·24·13	22 24	22 31	12 15	11 59	12 22	12 22
$-x^9$	$\frac{5}{2}\text{-}\frac{15}{2}$	$\bar{2}$·15·6	16 5	15 47	18 51	19 7	7 39	7 34
$-x^{10}$	$\frac{13}{7}\text{-}\frac{13}{4}$	$\bar{4}$·13·7	24 22	23 59	9 26	9 47	15 25	15 3
$-y^2$	$\frac{25}{9}\text{-}\frac{25}{7}$	$\bar{7}$·25·9	18 54	19 2	14 45	14 38	14 40	14 52
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] 32	19 29
$-y^3$	$24\text{-}\frac{24}{9}$	$\bar{9}$·24·1	18 45	19 5	25 34	25 27		

TRANSACTIONS

OF THE

CONNECTICUT ACADEMY

OF

ARTS AND SCIENCES.

VOLUME III.

NEW HAVEN:
PUBLISHED BY THE ACADEMY.
1874 to 1878.

Tuttle, Morehouse & Taylor, Printers, New Haven.

II. A List of Writings relating to the Method of Least Squares, with historical and critical Notes. By Mansfield Merriman.

The following list contains the titles of 408 papers, books and parts of books, relating to the Method of Least Squares and the Theory of accidental Errors of Observation, chronologically arranged according to their dates of publication. The first was issued in the year 1722 and the last in 1876. Previous to 1805, the year of Legendre's announcement of the principle of Least Squares, there are 22 titles; since 1805 there is a continual yearly increase in the number of titles,—thus:

From	1805	to 1814	inclusive,	there are	18	titles,
"	1815	" 1824	"	"	30	"
"	1825	" 1834	"	"	32	"
"	1835	" 1844	"	"	45	"
"	1845	" 1854	"	"	63	"
"	1855	" 1864	"	"	71	"
"	1865	" 1874	"	"	95	"

These books and memoirs are in eight languages, and classified according to the place of publication, they fall under twelve countries. It may be interesting to note the number belonging to each, viz:

Countries.		*Languages.*	
Germany	153	German	167
France	78	French	110
Great Britain	56	English	90
United States	34	Latin	16
Belgium	19	Italian	9
Russia	16	Dutch	7
Italy	14	Danish	5
Austria	10	Swedish	4
Switzerland	9		
Holland	7	Total	408
Sweden	7		
Denmark	5		
Total	408		

The 408 titles may be roughly classified as 313 memoirs, 72 books and 23 parts of books. They were written by 193 authors, 127 of whom produced only one book or paper each.

In preparing this list I have been able to use only the libraries of Yale College, (including that of the Connecticut Academy of Arts and Sciences,) but I think no work in those libraries relating to the subject has been left unconsulted. Out of the total 408 works, I have seen 312; the titles of these and the accompanying notes have been drawn from actual inspection. I have made no attempt to consult the literature of the Russian and Hungarian languages, and with wider library facilities the number of titles in the Italian, Dutch and Scandinavian languages would undoubtedly have been greater. Of course no work of this kind can be regarded as complete.

It has been my aim to record all writings which can be considered as contributions to the science of the Adjustment of Observations, and I think that those marked as actually inspected may be truly so regarded. Many works on Astronomy and Probability which devote but a page or two to the subject, as well as numerous practical papers, in which the Method of Least Squares is used incidentally and briefly, have been left unnoticed; to record all of these would be well nigh impossible, nor would the value of the list be thereby increased. Among the 96 which I have not seen there may possibly be a few that would be rejected after actual inspection.

The following is the arrangement of the list.

At the head of a title is placed the year of publication. In the case of memoirs this often differs from the date of the volume in which they are contained; for instance, Nautical Almanacs are published several years preceding and Transactions of Learned Societies often several years after the date which they bear. When a memoir is published in parts extending over two or more years it is recorded under the date of the first part.

The author's name follows the date of publication. In the index at the end of the list the full names of authors are given, and a distinction is made in the text when two persons of the same surname have written on the subject.

The titles of books, and pamphlets published as books, are printed in *italics*, and the titles of memoirs in ordinary type. Those which I have actually inspected have their titles enclosed in single quotation marks ('——') and these are intended to correspond with the originals in punctuation, spelling and when possible in the use of capitals.

The place of publication of books is given, with references to subsequent editions or translations. The usual terms 4to, 8vo, etc. are added, although they give little idea of the size of a book, and the number of pages is noted whenever I have been able to ascertain it. When a single chapter only of a book relates to the subject, the title of that chapter is treated like a memoir.

After the title of a memoir is placed *in italics* the title of the volume containing it. This is abridged in the usual manner, but the words of the title are never transposed. For instance, *Bulletin des Sciences de la Societè Philomathique de Paris* is abridged into *Bull. Soc. Philom. Paris.* In a few instances I have added the place of publication or have prefixed the name of the editor in order to ensure perfect clearness. The number of the volume and the pages which are devoted to the memoir are given; the mention of the year renders it unnecessary to note the various series.

When the work was begun it was intended to make the notes very full so as to give a tolerably complete history of the Method of Least Squares. But as the number of titles began to multiply under research it became evident that the plan would produce a manuscript too voluminous for publication. The notes were hence abbreviated into their present form. The work begun as historical has, I am afraid, ended by being largely bibliographical.

Sometimes the notes give an account of the contents of a memoir or an estimate of its value; sometimes they take the form of a direct quotation from the memoir itself or state the opinion of some subsequent reviewer; and occasionally they offer critical remarks of my own. But always they aim to give such cross references as will enable the student to follow up special lines of investigation and gain the fullest information concerning a particular memoir or book. Brief as the notes are, I hope they will be found at least suggestive by those who use them. To the future historians of mathematical science they will undoubtedly be of very great value.

The mode of cross reference usually adopted is to mention simply the year and author. Thus "1818 BESSEL" refers either to a book published in 1818 by BESSEL or to notes under that heading.

The following table points out some of the most valuable papers on the proofs of the Method of Least Squares:

First publication of the Method .... see	1805	LEGENDRE.
First and Second Proofs	1808	ADRAIN.
Third Proof	1809	GAUSS.
Fourth Proof	1810	LAPLACE.
Fifth Proof	1812	LAPLACE.
Theory and Practice compared	1818	BESSEL.
Sixth Proof	1823	GAUSS.

Seventh Proof .... see	1825	IVORY.
Eighth Proof	1826	IVORY.
Proof of the Arithmetical Mean	1832	ENCKE.
Ninth Proof	1837	HAGEN.
Tenth Proof	1838	BESSEL.
LAPLACE's Proof extended and improved	1844	ELLIS.
Eleventh Proof	1844	DONKIN.
ADRAIN's Second Proof rediscovered	1850	HERSCHEL.
Twelfth Proof	1856	DONKIN.
On the Arithmetical Mean	1864	DEMORGAN.
[Thirteenth Proof]	1870	CROFTON.
Analysis of several Proofs	1872	GLAISHER.
Proof of the Arithmetical Mean	1875	SCHIAPARELLI.

I have drawn information from every source within my reach. On the Proofs of the Method I have found GLAISHER's memoir of 1872 of the greatest value, and while working on the early writers TODHUNTER's *History of Probability* was continually before me. It is here also the place to acknowledge my indebtedness to Prof. H. A. NEWTON of Yale College for valuable suggestions and kind assistance.

1722 COTES. 'Æstimatio errorum in mixta mathesi, per variationes partium trianguli plani et sphærici.' *Opera miscellanea* (appended to *Harmonia mensurarum;* Cantabrigiæ, 4to), pp. 1–22. —Memoir republished, Lemgoviæ, 1768, 8vo.

Only the closing paragraph relates to accidental errors of observation; this gives the following rule: "Sit p locus Objecti alicujus ex Observatione prima definitus, q, r, s ejusdem Objecti loca ex Observationibus subsequentibus; sint insuper P, Q, R, S pondera reciproce proportionalia spatiis Evagationum, per quæ se diffundere possint Errores ex Observationibus singulis prodeuntes, quæque dantur ex datis Errorum Limitibus; & ad puncta p, q, r, s posita intelligantur pondera P, Q, R, S, & inveniatur eorum gravitatis centrum Z: dico punctum Z fore Locum Objecti maxime probabilem, qui pro vero ejus loco tutissime haberi potest."

COTES's rule only agrees with modern methods when the observations are directly made upon one quantity. See LAPLACE, *Théorie analytique des Probabilités*, third edition, p. cxxxviii, p. 346; and IVORY, *Phil. Mag.*, 1825, Vol. LXV, p. 4.

1749 EULER. *Pièce qui a remporté le prix de l'Academie royale des sciences en* 1748, *sur les inegalities du mouvement de Saturne et de Jupiter.* Paris, 4to.

Contains a method for the combination of linear equations similar to the following.

1750 MAYER. Abhandlung über die Umwälzung des Mondes um seine Axe. *Kosmographische Nachrichten u. Sammlungen* for 1748, pp. 52–183.

By twenty-seven observations upon the position of a moon spot MAYER obtained twenty-seven equations each containing three unknown quantities. To solve these he added together those nine equations in which the values of the coefficients of one of the unknown quantities were the greatest, then the nine in which these coefficients were the least and lastly the remaining nine; thus obtaining three resulting equations with three unknown quantities.

MAYER's method is given by LALANDE, *Astronomie*, second edition, Vol. III, pp. 418–428. See WOLF, *Handbuch der Math.*, Vol. I, p. 279, Vol. II, p. 199; and also below, 1830 FRANCOEUR.

[1754] KÄSTNER. Om geometriska aberrationer. *Svenska Vetensk. Acad. Handl.* for 1753, p. 126. —German translation in *Schwedische Akad. Abhandl.* for 1753, p. 131.

Treats of errors in Surveying and probably contains nothing of value on the theory of accidental errors. See also the same memoirs for 1768, p. 147 and p. 159.

1755 BOSCOVICH. *De littera expeditione per Pontificam ditionem ad dimetiendos duos meridiani gradus.* Romae, 4to, pp. xxii, 516. —French translation by HUGON entitled *Voyage astronomique et geographique....*; Paris, 1770, 4to.

A method of combination of observations for the determination of the mean ellipticity of the earth from measured arcs of the meridian is here used. The method itself is proved in the French translation; see below 1760. For description of the book see TODHUNTER, *History of Theories of Attraction....*, Vol. I, pp. 305–321, 332–334.

1756 SIMPSON. 'A letter to the Right Honourable George Earl of Macclesfield, President of the Royal Society, on the Advantage of taking the Mean of a number of Observations, in practical Astronomy.' *Phil. Trans. Lond.* for 1755, Vol. XLIX, Pt. I, pp. 82–93. —Reprint, see 1757.

This memoir is interesting and valuable as being the first in which the Theory of Probability is applied to the discussion of errors of observation and in which the idea of a law of facility of error is implied. At the beginning of the letter SIMPSON says that his attention had been called to the subject by the fact that "some persons, of considerable note, have been of opinion, and even publickly maintained, that one single observation, taken with due care, was as much to be relied on as the Mean of a great number."

The letter contains two propositions; the first gives a method of determining the probability that the error of the mean of n observations shall be less than an assigned value, provided it is equally probable that the error of a single observation may be any one of the quantities, $-v, \ldots -3, -2, -1, 0, 1, 2, 3, \ldots v$. The second gives a method of determining that probability, provided the probabilities of the single errors $-v, -v+1, \ldots -1, 0, 1, \ldots v-1, v$ are proportional to the terms $1, 2, \ldots v, v+1, v, \ldots 2, 1$. This is illustrated by a numerical example. SIMPSON remarks that the advantage of the mean can be shown, whatever series be used to express the chances of the errors.

1757 SIMPSON. 'An Attempt to show the Advantage arising by Taking the Mean of a Number of Observations, in practical Astronomy.' *Miscellaneous Tracts*....(London, 4to), pp. 64–75.

A reprint of the preceding, the opening and closing paragraphs being omitted and nearly four pages of new matter added. We find here the following axioms stated for the first time; 1. that positive and negative errors are equally probable, 2. that there are certain assignable limits within which all errors may be supposed to fall.

In the added matter the case of *continuous* errors is discussed, and the probability that the error of the mean is less than an assigned value found for the case of the second proposition by making v and other quantities in the formulæ infinite. SIMPSON represents the law of facility of error geometrically by the sides of an isosceles triangle and draws a curve to show the increased precision of the mean as compared with single observations. He closes by finding under the same supposition as to the law of facility, the probability that the mean is nearer to the truth than a single observation taken at random. The whole memoir must have been extremely valuable at the time of its publication.

1760 BOSCOVICH. De recentissimis graduum dimensionibus, et figura, ac magnitudine terrae inde derivanda. *Philosophia recentior á* BENEDICTO STAY....(Romae, 3 vols., 8vo), Vol. II, pp. 406–426.

A method of combining discordant observations upon the lengths of degrees of the earth's meridian is here given. The adjustment is effected under the two conditions that the sum of the negative errors shall be equal to the sum of the positive errors, and that each sum shall have the least possible value. The problem was solved by a geometric construction depending upon the properties of the centre of gravity of figures.

The method is also given in the French translation of BOSCOVICH's work of 1755. See LINDENAU in *Zach's Monatliche Correspondenz*, 1806, Vol. XIV, p. 132; TODHUNTER, *Hist. of Theories of Attraction*...., Vol. I, pp. 321–332; and below, 1785 BERNOULLI, 1792 LAPLACE.

1760 LAMBERT. *Photometria sive de mensura et gradibus luminis*..... Augustæ Vindelicorum, 8vo, pp. [xxx], 547.

Contains many remarks on the arithmetical mean and also proposes a method for judging of the precision of the measurements, which consists in comparing the mean with a new mean found after rejecting that observation deviating most from the first mean. See *Nova Acta Lipsiae*, 1760, p. 560; also *Lambert's Beyträge*..... Vol. I, p. 426.

1765 LAMBERT. 'Theorie der Zuverlässigkeit der Beobachtungen und Versuche.' *Beyträge zum Gebrauche der Mathematik*.... (Berlin, 1765–70, 3 vols., 8vo; second ed., 1792), Vol. I, pp. 424–488.

Contains a method of adjusting simple observations founded on the principle that the algebraic sum of the errors shall be zero. The method is illustrated by the determination of empirical formulæ for the length of the seconds pendulum, the declination of the magnetic needle, etc.

1774 LAPLACE. 'Determiner le milieu que l'on doit prendre entre trois observations données d'un même phénomène.' *Mém. Acad. Paris, par divers savans* [étrangers], Vol. IV, pp. 634–644.

This is Part V of the 'Mémoire sur la probabilité des causes par les évènemens,' which occupies pages 621–656 of the volume. It contains the first attempt to deduce a rule for the combination of observations from the principles of Probability.

LAPLACE begins by saying that the law of probability of errors of observation may be represented by a curve whose equation is $y=\varphi(x)$, x being any error and y its probability; and this curve must have three properties: 1st, it must be symmetrical with reference to the axis of y, since positive and negative errors are equally probable; 2nd, the axis of x must be an asymptote, since the probability of the error ∞ is 0; 3rd, the area of the curve must be unity, since it is certain that an error will be committed.

LAPLACE takes $\varphi(x)$ as $\varphi(x)=\frac{1}{2}me^{-mx}$ (x being regarded as always positive), but his reasons for doing so are slight. With this law he finds the mean of three observations, regarding it as corresponding to an ordinate which divides a curve $u=\varphi(x_1)\varphi(x_2)\varphi(x_3)$ into two equal parts. His result is as follows: Let M_1, M_2 and M_3 be the three measurements, of which M_1 is the least; let M_1+x be the mean; it is required to find x. Put $M_2-M_1=p$ and $M_3-M_2=q$: then x is given by

$$x=p+\frac{1}{m}\log_e(1+\tfrac{1}{3}e^{-mp}-\tfrac{1}{3}e^{-mq}),$$

in which m is a constant depending upon the precision of the observation. LAPLACE then shows that this value cannot agree with the rule of the arithmetical mean, and he computes a table for finding x for certain given ratios of q to p. For instance

if $q=0.0\,p$ .... $x=0.860\,p$,
$q=0.1\,p$ .... $x=0.894\,p$,
$q=0.2\,p$ .... $x=0.916\,p$,
$q=0.3\,p$ .... $x=0.932\,p$,
. . . .

Thus if three measurements of an angle give

$$M_1 = a^\circ\, b'\, 0'', \quad M_2 = a^\circ\, b'\, 40'' \quad \text{and} \quad M_3 = a^\circ\, b'\, 50'',$$

we have $p=40''$, $q=10''$ and $q=0.25p$; then from the table $x=37''$ and the adjusted result is $a^\circ\, b'\, 37''$, while by the usual rule of the average we would have $a^\circ\, b'\, 30''$. By LAPLACE's table the mean will lie nearer to the two observations which most nearly agree, than in the common method.

For remarks on this memoir see TODHUNTER, *History of Probability*, p. 469, and GLAISHER, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, pp. 121–123.

[1774] LAGRANGE. 'Mémoire sur l'utilité de la méthode de prendre le milieu les résultats de plusieurs observations; dans lequel on examine les avantages de cette méthode par le calcul des probabilités; & où l'on résoud différens problêmes relatifs à cette matière.' *Miscellanea Taurinensia* (*Mél. Soc. Turin*) for 1770–1773, Vol. V, pp. 167–232 of the math. part.

This memoir is a more thorough presentation of the subject treated by SIMPSON in 1757 with much new matter added. LAGRANGE makes no allusion however to previous writings on the subject. The expression "Law of Facility of Error" occurs here for the first time.

For an extended account of the contents of the memoir see TODHUNTER, *History of Probability*, pp. 301–313. See also below 1785 BERNOULLI, 1788 EULER, and 1804 TREMBLEY. In 1850 ENCKE gives a translation of part of the memoir, with comments.

1778 BERNOULLI (Daniel). 'Dijudicatio maxime probabilis plurium observationum discrepantium atque verisimillima inductio inde formanda.' *Acta Acad. Petrop.* for 1777, Pt. I, pp. 3–23 of the memoirs.

The now familiar illustration of a marksman firing at a target is here introduced, and the conclusion drawn that small errors are more probable than large ones, and that the method of taking the arithmetical mean "non sine ratione dubitare potest," since it supposes the observations of equal weight.

DANIEL BERNOULLI takes a circle $y=\sqrt{r^2-x^2}$ as representing the law of facility of error, y being proportional to the probability of the error x, and r a constant. Then if observations give the errors x_1, x_2, x_3 …., the product $\sqrt{r^2-x_1^2}\,\sqrt{r^2-x_2^2}\,\sqrt{r^2-x_3^2}$ must be a maximum to give the most probable value of the observed quantity. He finds that this value coincides with that given by the rule of the arithmetical mean for one and for two observations, and that it nearly coincides for three when a suitable value is given to r. For a greater number than three his method leads to unmanageable equations. He closes by remarking that the problem is indeterminate. See *Zach's Monatliche Correspondenz*, 1805, Vol. XI, pp. 486–490.

DANIEL BERNOULLI's method agrees closely with modern theory. See 1778 EULER, 1785 BERNOULLI and TODHUNTER, *History of Probability*, p. 236.

1778 EULER. 'Observationes in praecedentem dissertationem Illustr. BERNOULLI.' *Acta Acad. Petrop.* for 1777, Part I, pp. 24–33 of the memoirs.

EULER considers that DANIEL BERNOULLI is correct in objecting to the arithmetical mean when the observations are of unequal precision, but that he "was quite arbitrary in proposing to make the product of the probabilities a maximum. EULER proposes another method which amounts to making the sum of the fourth powers of the probabilities a maximum."—TODHUNTER, *Hist. of Probability*, p. 237.

1781 LAPLACE. 'Mémoire sur les Probabilités.' *Hist. Acad. Paris* for 1778, pp. 227–332 of the math. part.

Pages 322–332 are devoted to the discussion of the mean to be taken between discordant observations. LAPLACE says that by the expression *mean* or *mean result* an infinite number of things may be understood. Of these the one which is implied in adjusting observations is a value such that the resulting error shall be a minimum, and this corresponds to a value such that the sum of the errors, each multiplied by its probability shall be a minimum. The method of his memoir of 1774 is then substantially repeated.

LAPLACE remarks that if $\pm a$ be the limits of error, the law of facility ought to be $\varphi(x)=\frac{1}{2a}\log_\varepsilon \frac{a}{x}$. His methods lead to unmanageable equations.

1785 BERNOULLI (Jean, III). 'Milieu.' Art. in the *Encyclopedie Méthodique* (Paris, 4to), Vol. II, pp. 404–409. —Second edition, 1789, entitled *Dictionaire Encycl. des Mathématiques.*

BOSCOVICH's method of 1760 and LAMBERT's of 1765 are referred to, and an account is given of DANIEL BERNOULLI's memoir of 1778, which differs slightly from the memoir itself. An account of LAGRANGE's memoir of 1774 is presented with considerable fullness. See TODHUNTER, *Hist. of Probability*, p. 442.

1788 EULER. 'Eclaircissemens sur le mémoire de Mr DE LA GRANGE inséré dans le V^{e} volume de *Mélanges de Turin*, concernant la méthode de prendre le milieu entre les résultats de plusieurs observations, &c.' *Nova Acta Acad. Petrop.* for 1785, Vol. III, pp. 289–297 of the memoirs.

The memoir seems to have no value; See TODHUNTER, *Hist. of Probability*, p. 250.

1789 LEGENDRE. 'Mémoire sur les Opérations trigonométriques, dont les résultats dépendent de la figure de la terre.' *Hist. Acad. Paris* for 1787, pp. 352–383 of the memoirs.

The method of *indeterminate corrections*, afterwards of so much use in the computations of Least Squares is here stated and used, although perhaps not for the first time. See 1820 LEGENDRE.

1792 LAPLACE. 'Sur les degrés mesurés des méridiens, et sur les longueurs observées du pendule.' *Hist. Acad. Paris* for 1789, pp. 18–43 of the memoirs.

The matter of this memoir is mostly reproduced in Sections 39–42 of Chap. V, Book III, of the *Traité de mécanique céleste*.

First a method is presented of determining an elliptic meridian so that the greatest error shall be a minimum; see 1831 CAUCHY. Secondly a method is developed for finding an ellipse subject to the following conditions: "1° que la somme des erreurs soit nulle; 2° que la somme des erreurs prises toutes avec le signe + soit un minimum." LAPLACE mentions that BOSCOVICH (1760) had solved the same problem by a different method. He alludes to the ellipse thus determined as "l'ellipse la plus probable."

1799 KRAMP. 'Table première. Intégrales de $e^{-tt}dt$, depuis une valeur quelconque de t jusqu' à t infinie.' and 'Table seconde. Logarithmes des intégrales $\int e^{-tt}dt$.' *Analyse des refractions* . . . (Strasbourg, 4to), pp. 195–206.

For description of these tables, afterwards of so great use in the Theory of Errors, see GLAISHER, *Lond. Phil. Mag.*, 1871, Vol. XLII, p. 431.

1804 TREMBLEY. 'Observations sur la méthode de prendre les milieux entre les observations.' *Mémoires Acad. Berlin* for 1801, pp. 29–58 of the math. part.

"Appears to be of no value whatever."—TODHUNTER, *Hist. of Probability*, p. 428.

1804 PRONY. *Recherches physico-mathématiques sur la théorie des eaux courantes.* Paris, 4to.

Contains applications of LAPLACE's methods of 1792 to the deduction of empirical formulæ from discordant observations.

1805 LEGENDRE. '*Nouvelles méthodes pour la détermination des orbites des comètes.*' Paris, 4to, pp. viii, 80. —Second edition, see 1806.

The date 1806 is generally given for this book, in which year a second edition seems to have been issued. The copy before me is plainly dated, "An XIII—1805."

In the preface LEGENDRE gives an outline of his method for computing orbits. On page viii he says: "Il faut ensiuite, lorsque toutes les conditions du problême sont exprimées convenablement, déterminer les coëfficiens de manière à rendre les erreurs les plus petites qu'il est possible. Pour cet effet, la méthode qui me paroît la plus simple et la plus générale, consiste à rendre *minimum* la somme des quarrés des erreurs. On obtinent ainsi autant d'equations qu'il y a de coëfficiens inconnues; ce qui achève déterminer tous les élémens de l'orbite.....la méthode dont je viens de parler, et que j'applle *Méthode des moindres quarrés*, peut être d'une grande utilité....."

On page 64 is an application of the method to the solution of three equations with two unknown quantities, in which the now well-known rule for the solution of normal equations is followed. On page 68 and 69 are references to its use. Pages 72–80 constitute an Appendix "Sur la Méthode des moindres quarrés." In this, after having mentioned that it is impossible that the sum of the errors should be zero when the number of given equations exceeds that of the unknown quantities, LEGENDRE says: "De tous les principes ....je pense qu'il n'en est pas de plus général, de plus exact, ni d'une application plus facile que celui .... qui consiste à rendre *minimum* la somme des quarrés des erreurs. Par ce moyen, il s'établit entre les erreurs une sorte d'équilibre qui empêchant les extrêmes de prévaloir, est tres-propre à faire connoître l'état du systême le plus proche de la vérité."

LEGENDRE then proceeds to deduce the rule for the formation of "l'equation du *minimum* par rapport à l'une des inconnues," or as we now say of a normal equation. His notation is the following: in the n equations

$$
\begin{aligned}
0 &= a + bx + cy + \ldots \\
0 &= a' + b'x + c'y + \ldots \\
0 &= a'' + b''x + c''y + \ldots \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

$a, b, c \ldots, a', b', c' \ldots$ are known by observation or theory, and $x, y, \ldots$ are to be determined. By forming the sum of the squares of these equations, differentiating with reference to each unknown separately and placing the derivatives equal to zero, he finds

$$
\begin{aligned}
0 &= \textstyle\int ab + x\int b^2 + y\int bc + \ldots \\
0 &= \textstyle\int ac + x\int bc + y\int c^2 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

which are the same in number as the unknown quantities $x, y \ldots$, and in which

$$
\begin{aligned}
\textstyle\int ab &= ab + a'b' + a''b'' + \ldots \\
\textstyle\int b^2 &= b^2 + b'^2 + b''^2 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

LEGENDRE next demonstrates that the rule of the arithmetical mean is a particular case of his general principle. He then supposes the position of a point in space to be determined by three observations and finds the values of its coördinates given by the method. Noticing their identity with those for the centre of gravity of three points in space, he announces that the sum of the squares of the distances of all the molecules of a body from its centre of gravity must be a minimum, and that hence "la méthode des moindres quarrés fait connoître, en quelque sorte, le centre autour duquel viennent se ranger tous les résultats fournis par l'expérience, de manière à s'en écarter le moins qu'il est possible." A numerical example of the application of the method to the determination from five observations of the form of a meridian of the earth closes the book.

The honor of the first publication of the Method of Least Squares belongs to LEGENDRE. Although he failed to claim that his method gives most probable or most advantageous results, yet the remarks above quoted indicate that he fully recognized it as a rule giving a plausible and reasonable mean. See below 1814.

1805 PUISSANT. *Traité de Géodésie, ou Exposition des Méthodes astronomiques et trigonométriques*..... Paris, 4to. —Third edition, see 1842.

This contains an explanation and application of LEGENDRE's Method of Least Squares. See review in *Zach's Monatliche Correspondenz*, Vol. XVI, p. 455.

1806 LEGENDRE. 'Supplément aux Nouvelles Méthodes pour la détermination des orbites des comètes.' Paris, 4to, pp. 55.

The copy of this supplement which is before me is preceded by a title page reading, "*Nouvelles méthodes pour la détermination des orbites des comètes; avec un Supplément*..... A Paris, . .. Année 1806." This title page was probably prefixed to copies of the work of 1805, with the Supplement added, as the date 1806 is usually stated for it. A second Supplement appeared in [1820].

On pages 28 and 43 the Method of Least Squares is used.

1806 LINDENAU. 'Ueber den Gebrauch der Gradmessungen zur Bestimmung der Gestalt der Erde.' *Zach's Monatliche Correspondenz*, Vol. XIV, pp.113–158, etc.

LAPLACE's method of 1792 is stated and used in the discussion of the most probable elliptic meridian. The method is attributed to BOSCOVICH; see 1760.

On pages 138–143 "die....von LEGENDRE vorgeschlagenen Methode, die er *Méthode des moindres quarrés* nennt" is explained and applied in the determination of the elliptic meridian. The date of LEGENDRE's *Nouvelles méthodes*....is given as 1806.

1806 GAUSS. 'II Comet vom Jahr 1805.' *Zach's Monatl. Corres.*, Vol. XIV, pp. 181–186.

On page 184 GAUSS says: "LEGENDRE's Werk....habe ich noch nicht gesehen. Ich hatte mit Fleiss mir deswegen keine Mühe gegeben, um bey der Arbeit an meiner Methode ganz in der Kette meiner eigenen Ideen zu bleiben. Durch ein paar Worte, die DE LA LANDE in der letztern *Historie de l'Astronomie*, 1805, fallen lässt, *méthode des moindres quarrés*, gerathe ich auf die Vermuthung, dass ein Grundsatz, dessen ich mich schon seit zwölf Jahren bey mancherley Rechnungen bedient habe, und den ich auch in meinem Werke mit gebrauchen werde, ob er wol zu meiner Methode eben nicht wesentlich gehört,—dass dieser Grundsatz auch von LEGENDRE benutzt ist."

The work of LEGENDRE here alluded to is the *Nouvelles méthodes* ....1805, and that of GAUSS the *Theoria motus* ...1809, then in preparation. GAUSS mentions some of the advantages of his method for computing orbits but gives no hint of the principle of Least Squares.

1808 BOWDITCH. 'Solution of Mr. PATTERSON's Prize Question for correcting a survey, proposed in No. II. page 42, No. III. page 68, by NATHANIEL BOWDITCH, to whom the Editor has awarded the prize of ten dollars.' *The Analyst or Math. Museum*, Vol. I, pp. 88–92.

The Prize Question was: "In order to find the content of a piece of ground, ...I measured, with a common circumferentor and chain, the bearings and lengths of its several sides,..... But upon casting up the difference of latitude and departure, I discovered....that some error had been contracted in taking the dimensions. Now it is required to compute the area of this enclosure, on the *most probable supposition* of this error."

BOWDITCH's solution depends on several "principles" or hypotheses, the chief of which is "that in measuring the lengths of any lines the errors would probably be in proportion to their lengths." No principles of the Theory of Probability are employed except such as are by common sense implied. His solution coincides with that given by the Method of Least Squares.

This Prize Question undoubtedly led to the following 'Research' by ADRAIN, the Editor of *The Analyst*.

1808 ADRAIN. 'Research concerning the probabilities of the errors which happen in making observations.' *The Analyst or Math. Museum*, Vol. I, pp. 93–109.

This paper seems to have been unknown to mathematicians until 1871 when it was partly reprinted in *Amer. Jour. Sci.*, see 1871 ABBE. It is of great historical interest as containing the first deduction of the law of facility of error

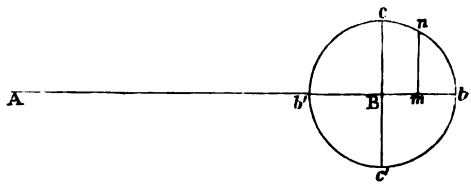
$$\varphi(x) = ce^{-h^2x^2}$$

$\varphi(x)$ being the probability of any error x, and c and h constants depending upon the precision of the measurement. The term "Least Squares" is not used, and ADRAIN seems to have been entirely unacquainted with LEGENDRE's writings.

ADRAIN gives two deductions of this law. The first, occupying pages 93–95 has been reprinted as noted above and need not here be repeated. It depends upon the "self-evident principle" that the true errors of measured quantities are proportional to the quantities themselves. The arbitrary nature of this assumption is shown by GLAISHER, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, pp. 75–81, where the proof is analyzed and regarded as "very slight and inconclusive."

The second proof occupies pages 96–97, and not having been alluded to in the reprint noted above, I give it in ADRAIN's own words as *The Analyst* is quite rare:

"Suppose that the length and bearing of AB are to be measured; and that the little equal straight lines Bb, Bc are the equal probable errors, the one $Bb=Bb'$ of the length of AB, and the other $Bc=Bc'$ (perpendicular to the former) of the angle at A, when measured on a circular arc to the radius AB: and let the question be to find such a curve passing through the four points b, c, b', c', which are equally distant from B, that, supposing the measurement to commence at A, the probability of terminating on any point of the curve may be the same as the probability of terminating on any one of the four points b, c, b', c'."



Then follows trivial reasoning which ends by concluding that "the curve must be the simplest possible" and "must consequently be the circumference of a circle having its centre in B." This established, the proof is the following:

"Now let us investigate the probability of the error $Bm=x$, and of $mn=y$. Let X and Y be two similar functions of x and y denoting those probabilities, X', Y', their logarithms, then $X\times Y=$ constant, or $X'+Y'=$ constant, and therefore $\dot{X}'+\dot{Y}'=0$, or $X''\dot{x}+Y''\dot{y}=0$, whence $X''\dot{x}=-X''\dot{y}$. But $x^2+y^2=r^2=Bb^2$, therefore $x\dot{x}=-y\dot{y}$, by which dividing $X''\dot{x}=-Y''\dot{y}$, we have $\frac{X''}{x}=\frac{Y''}{y}$; and therefore, by a fundamental principle of similar functions, the similar functions $\frac{X''}{x}$ and $\frac{Y''}{y}$ must be each a constant quantity: put then $\frac{X''}{x}=n$, and we have $X''\dot{x}=nx\dot{x}$, that is $\dot{X}'=nx\dot{x}$, and the fluent is $X'=c+\frac{nx^2}{2}$; in like manner we find $Y'=c+\frac{ny^2}{2}$ and therefore the probabilities themselves are $e^{c+\frac{nx^2}{2}}$ and $e^{c+\frac{ny^2}{2}}$, in which n ought to be negative, for the probability of x grows less as x grows greater."

I have seen no allusion to this proof by any subsequent writer. It is essentially the same as given in 1850 by HERSCHEL and usually called HERSCHEL's proof. I regard it as defective in taking "$X\times Y=$ constant," or in considering the probabilities of the x and y deviations as independent. See 1850 ELLIS. See BOOLE's *Finite Differences* (Cambridge, 1860), pp. 228–229.

In his first proof ADRAIN had found for the probability of the error x in the observed value a the expression $e^{c+\frac{mx^2}{2a}}$, and had shown that the most probable values of the observed quantities $a, b, c, \ldots.$ whose errors are $x, y, z, \ldots$ must satisfy the condition

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\ldots. = \text{a minimum.}$$

This principle he applies to four problems; the first showing that the arithmetical mean is a particular case of the method, the second to determine the most probable position of an observed point in space, which is shown to be "precisely in the centre of gravity of all the given points;" the third "to correct the dead reckoning at sea," and the fourth "to correct a survey."

As remarked by ABBE "we must credit Dr. ADRAIN with the independent invention and application of the most valuable arithmetical process that has been invoked to aid the progress of the exact sciences."—*Amer. Jour. Sci.*, 1871, Vol. I, p. 415.

1809 GAUSS. 'Determinatio orbitæ observationibus quotcunque maxime satisfacientis.' *Theoria motus corporum coelestium....* (Hamburgi, 4to), Lib. II, Sect. III, pp. 205–224. —French translation, see 1855 BERTRAND. —English translation by DAVIS (Boston, 1858, 4to), pp. 249–273. —German trans. by HAASE (Hannover, 1865).

That demonstration of the Method of Least Squares usually called GAUSS's proof or GAUSS's first proof is here presented. Assuming that the arithmetical mean of direct observations is the most probable value of the measured quantity, it deduces that the law of facility of error is given by

$$\varphi(x) = ce^{-h^2x^2}$$

from which the principle of Least Squares at once follows. This proof has been adopted by the majority of books on the subject; see for instance 1832 ENCKE, 1857 DIENGER, 1858 RITTER, 1864 CHAUVENET, 1867 HANSEN and MERRIMAN's *Elements of the Method of Least Squares* (London, 1877, 8vo).

The demonstration as given by GAUSS contains three defects. 1. It is not recognized that the probability of a definite error x, is an infinitesimal; this is avoided by some later writers. 2. The distinction between *true errors* and *residuals* (or calculated errors) is not sharply drawn; according to GAUSS's reasoning the law $\varphi(x)=ce^{-h^2x^2}$ is not strictly a "law of facility of error" but only a law of distribution of residuals. 3. The rule of the arithmetical mean is assumed. For critical analyses of this proof see below, 1843 REUSCHLE, 1844 ELLIS and 1872 GLAISHER.

Practical features of the method,—the formation of normal equations and the determination of weights and degrees of precision are also discussed in the Section and hints are given regarding its use in astronomy. On page 221 is an attempt to justify the principle of

Least Squares on the ground that any other process would lead to impracticable calculations.

I quote a historical remark on page 221: "Ceterum principium nostrum, quo jam inde ab anno 1795 usi sumus, nuper etiam a clar. LEGENDRE in opere *Nouvelles méthodes ...comètes*, 1806 prolatum est....."

1810 BESSEL. *Untersuchungen über die scheinbare und wahre Bahn der grossen Cometen von* 1807. Königsberg, 4to.

The Method of Least Squares is used in determining the orbit and is called "den *moindres quarrés.*" See *Zach's Monatl. Corres.*, 1810, Vol. XXII, pp. 205–212.

1810 LAPLACE. 'Mémoire sur les approximations des formules qui sont fonctions de très-grands nombres, et sur leur application aux probabilités.' *Mém. Inst. France* for 1809, pp. 353–415, 559–565.

Pages 383–389 and 559–565 are devoted to the Theory of Errors, and the principle of Least Squares is proved "lorsque les résultats entre lesquels on doit prendre un milieu, sont donnés chacun par un très-grand nombre d'observations, quelles que soient d'ailleurs les lois de facilité des erreurs de ces observations."

The matter of these pages is reproduced in the *Théorie analytique des Probabilités*, pp. 329–335, 340–342; see below under 1812 for remarks concerning the proof. See *Bull. Soc. Philom. Paris*, 1810, Vol. II, pp. 132–136 for an account of the contents of the memoir.

1811 GAUSS. 'Disquisito de elementis ellipticis Palladis ex oppositionibus 1803, 1804, 1805, 1807, 1808, 1809.' *Comment. Soc. Göttingen*, Vol. I, 26 pp. —French translation, see 1855 BERTRAND.

Twelve equations involving six unknown corrections to the elements of the orbit are solved by the Method of Least Squares. We here find for the first time the notation

$$[ab]=a'\,b'+a''\,b''+a'''\,b'''+\ldots.$$

and also the algorithm for the solution of normal equations by successive substitution, since universally followed in lengthy computations.

A partial translation of the above with comments is given by ZACH in the *Monatl. Corres.*, 1811, Vol. XXIV, pp. 449–465.

1811 LAPLACE. 'Du milieu qu'il faut choisir entre les résultats des observations.' *Mém. Inst. France* for 1810, Pt. I, pp. 317–347.

Contains "the theory of errors substantially coincident with so much of the same theory as we find in pages 314–328 and 340–342 of the *Théorie....des Prob.*"—TODHUNTER, *History of Probability*, p. 490. See below 1812.

An account of the contents of this memoir, with historical remarks on the subject of Least Squares, is given by DELAMBRE, *Bull. Soc. Philom. Paris*, 1811, Vol. II, pp. 262–266, an English translation of which is in *Tilloch's Phil. Mag.*, 1812, Vol. XXXIX, pp. 240–244.

1811 LAPLACE. '*Du milieu qu'il faut choisir entre les résultats d'un grand nombre d'observations.*' *Connaissance des Tems* for 1813, pp. 213–223. —German translation in *Zach's Monatl. Corres.*, 1812, Vol. XXV, pp. 105–120.

Contains matter which is reproduced in the *Théorie . . . des Prob.*, 1812, pp. 322–329.

1812 LAPLACE. '*Théorie analytique des Probabilités.*' Paris, 4to. pp. 464. —Third edition, 1820, 4to, pp. cxlii, 506, with Introduction and three Supplements (see 1814, 1815, 1818, 1820). —Fourth edition (Vol. VII of *Oeuvres de Laplace*) Paris, 1847, 4to, pp. cxcv, 691.

". . . . the greater part of the *Théorie des Probabilités* is a reprint of papers in the Memoirs of the Academy, which appear to contain the contents of the first papers on which he set down his processes. These with preliminary chapters, descriptive not of what follows, but of the general methods which he drew from the following parts, make up the whole work."—DE MORGAN, *Theory of Probabilities* in *Encyc.. Metrop.*, p. 453. It "is by very much the most difficult mathematical work we have met with."—*Ibid*, p. 418. TODHUNTER in his *History of Probability*, p. 560, ELLIS (1844) and other writers have also testified to the abstruseness of LAPLACE's methods.

The Method of Least Squares is developed in Chap. IV, (pages 304–348) of the second part of the work. The analysis only extends to the case of two unknown quantities or elements, and the number of observations is required to be very large or infinite. Under these restrictions the Method is shown to give most advantageous results, whatever be the law of facility of error provided only that positive and negative errors are equally probable. LAPLACE's definition of *most advantageous results* is the following: " . . . . si l'on multiplie les erreurs possibles d'un élément par leurs probabilités respectives, le système le plus avantageux sera celui dans lequel la somme de ces produits tous pris positivement, est un *minimum*." The results thus obtained are not necessarily the *most probable*.

In the concluding paragraph of Chap. IV and in the opening pages of the First Supplement (see 1815), LAPLACE has given a general account of his method of analysis. These remarks and the table of contents at the end of the volume, give a much clearer idea of the steps of the demonstration than does Chap. IV itself. The principal objection against the validity of the proof is that it requires an infinite or very large number of observations. With this requirement, however, GAUSS's proof of 1809 becomes perfectly logical and the results are the *most probable*, not merely *most advantageous*.

LAPLACE's proof has been greatly improved by subsequent writers. ELLIS in 1844 extended it to any number of unknown quantities, TODHUNTER in his *History of Probability*, pp. 560–588 supplied a valuable commentary, and GLAISHER in 1872 presented it in a clear and simple form. See also below, 1824 POISSON, 1847 DEMORGAN, 1852 BIENAYMÉ, 1861 AIRY, 1873 LAURENT and 1875 DIENGER.

On pages 318–319 is given what is sometimes called LAPLACE's second proof of the Method of Least Squares. This depends on the definition that "la valeur moyenne de l'erreur a craindre en plus" should be a minimum. The reasoning is similar to GAUSS's proof of 1823; see ELLIS's and GLAISHER's papers quoted below under 1844 and 1872.

[1813] PLANA. Sur divers problêmes de probabilité. *Mém. Acad. Turin* for 1811–12, Vol. XX, pp. 355–408.

1814 CAUCHY. Mémoire sur le système de valeurs qu'il faut attribuer á divers Élemens, determineés par un grand nombre d'observations. [Paris, Lith. MS.]

Probably the same as his memoir of 1831.

1814 LAPLACE. '*Essai philosophique sur les probabilités.*' Paris, 4to. —Sixth edition, Paris, 1840, 8vo, pp. 274. —Introduction to second edition of *Théorie....des Prob.*, pp. cvi; to third edition, pp. cxlii; to fourth edition, pp. v–clxix.

Near the end are some remarks concerning the history of the Theory of Errors of observation, and descriptive of LAPLACE's processes.

1814 LEGENDRE. 'Méthode des moindres quarrés, pour trouver le milieu le plus probable entre les résultats de differentes observations.' *Mém. Inst. France* for 1810, Pt. II, pp. 149–154.

Pages 72–75 of the *Nouvelles méthodes*....1805, are here quoted and reference made to the practical applications of the method given in that work, in order to call attention to LEGENDRE's priority of publication.

1815 BESSEL. 'Ueber den Ort des Polarsterns.' *Berlin. Astron. Jahrb.* for 1818, pp. 233–240.

We here find the first mention of *probable error*. After giving 48 observations on the right ascension of Polaris whose arithmetical mean is $55^{m}\ 48^{s}.5104$, BESSEL says: "Der *wahrscheinliche* Fehler einer einzelnen Beobachtung ist, nach den wirklichen vorkommenden Fehlern zu urtheilen $=1^{s}.067$, und daher der wahrscheinliche Fehler des Endresultats $=0^{s}.154$. Die Grund dieser Schätzung des wahrscheinlichen Fehlers, beruhen auf der von GAUSS gegebenen Entwickelung der Wahrscheinlichheit, einen Fehler von gegebener

Grösse zu begehen; ihre Mittheilung muss ich bis auf eine andere Gelegenheit versparen." See 1816 BESSEL, and also page 196 of the memoir 1816 GAUSS.

1815 LAPLACE. 'Sur l'application du calcul des probabilités à la philosophie naturelle.' *Connaissance des Tems* for 1818, pp. 361–381. —First Supplement to third edition of *Théorie....des Prob.* (Paris, 1820, 4to), pp. 3–26.

This is devoted partly to a general description of LAPLACE's proof of the Method of Least Squares, and partly to the discussion of the probability of results obtained by that method; a numerical example illustrates the use of his formulæ. See TODHUNTER, *Hist. of Probability*, p. 610; also see 1869 TODHUNTER.

1816 BEECK-CALKOEN. Over de Theorie der Gemiddelde Waardij. *Verhandl. Nederland. Inst.*, Vol. II, pp. 1–19.

Treats of LAPLACE's method of adjustment of 1792.

1816 BESSEL. 'Untersuchungen über die Bahn des OLBERSschen Kometen.' *Abhandl. Akad. Berlin* for 1812–13, pp. 117–160 of the math. section.

BESSEL defines the *probable error* as follows: "Ich verstehe unter dieser Benennung die Grenze, die eine Anzahl kleinerer Fehler von einer *gleichen* Anzahl grösserer trennt, so dass es wahrscheinlicher ist, eine Beobachtung innerhalb jeder *weiteren* Grenze von der Wahrheit abirren zu sehen, als ausserhalb derselben." If we designate by $\frac{\Sigma x}{n}$ the mean of the errors all taken positively, by $\frac{\Sigma x^2}{n}$ the mean of the squares of those errors, and by r the probable error of a single observation, his demonstration shows that

$$r = 0.8453\,\frac{\Sigma x}{n} \quad \text{or} \quad r = 0.6745\sqrt{\frac{\Sigma x^2}{n}}.$$

BESSEL does not distinguish between true errors and residuals. These formulæ he uses in finding the probable errors of the elements of the orbits, which are deduced by the help of the Method of Least Squares.

1816 GAUSS. 'Bestimmung der Genauigkeit der Beobachtungen.' *Zeitschr. f. Astron. u. ver. Wiss.*, Vol. I, pp. 185–196. —Also *Gauss Werke*, Vol. IV (Göttingen, 1873, 4to), pp. 109–117. —See 1855 BERTRAND.

This memoir gives three methods for finding the probable error from given observations. The first, which is that usually presented in text-books, finds that r the probable error of an observation of the weight unity, is given (most probably) by

$$r = 0.6744897\sqrt{\frac{\Sigma x^2}{n}},$$

Σx^2 being the sum of the squares of the errors and n the number of observations, and that it is an even wager that the true value of r lies between

$$\frac{0.6744897\sqrt{\frac{\Sigma x^2}{n}}}{1-\frac{0.4769363}{\sqrt{n}}} \quad \text{and} \quad \frac{0.6744897\sqrt{\frac{\Sigma x^2}{n}}}{1+\frac{0.4769363}{\sqrt{n}}}$$

In the second method the most probable value of the sum Σx^m is discussed and formulæ for probable error found when m has the values 1, 2, 3, 4, 5 and 6. The second of these, which agrees with the one given above, is shown to be the best. The third method leads to a different and less accurate formula.

Nothing in the investigation shows whether Σx^2 is the sum of the squares of the *true* or of the *computed* errors. By later writers it has been generally taken as referring to the former. See 1816 Bessel, 1819 Young, 1823 Gauss, 1856 Peters, 1866 Börsch.

1818 Adrain. 'Investigation of the Figure of the Earth and of the Gravity in different Latitudes.' *Trans. Amer. Phil. Soc.*, Vol. I, pp. 119–135.

A formula for the length of the seconds pendulum is determined by the Method of Least Squares. Adrain alludes to the process as having been discovered by himself in 1808. See *Amer. Jour. Sci.*, 1871, Vol. I, p. 415, and *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, p. 78.

1818 Bessel. '*Fundamenta astronomiæ pro anno MDCCLV deducta ex observationibus viri incomparabilis* James Bradley *in specula astronomia Grenovicensi per annos* 1750–1762 *institutis.*' Regiomonti, folio, pp. 325.

In pages 18–21 results of the computations of the mean and probable errors of the declination and right ascension of certain stars as deduced from the observations are given. Three sets of measurements, two of 300 and one of 470, are investigated as a test of the exponential law $\varphi(x)=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$, the theoretical number of errors between given limits being computed from Kramp's tables and compared with the actual number of residuals. A close agreement was found, and this may perhaps be called a practical proof of the principle of Least Squares.

Tables of logarithms of $e^{t^2}\int_t^\infty e^{-t^2}dt$ are given; *regula minimorum quadratorum* is several times applied; and in pages 116–123 methods are given for finding probable errors of quantities indirectly observed.

1818 CAUCHY. Sur la méthode d'erreurs d'un grand nombre d'observations. [Paris], 4to.

1818 LAPLACE. 'Application du calcul des probabilités aux opérations géodésiques.' *Connais. des Tems* for 1820, pp. 422–440.

The matter of this article is reproduced in pages 3–25 of the Second Supplement to the third edition of the *Théorie des .... Prob.* For abstract see *Annal. Chem. et Phys.* 1817, Vol. V, pp. 351–356; also *Zeitschr. für Astron.*, Vol. V, pp. 1–9.

1818 LAPLACE. '*Deuxième supplément à la Théorie Analytique des Probabilités.*' Paris, 4to, pp. 50. —Fourth edition of *Théorie* ...., pp. 569–623.

"LAPLACE shows how the knowledge obtained from measuring a *base of verification* may be used to correct the values of the elements of the triangles of a survey ..... LAPLACE explains a method of treating observations which he calls the *method of situation* and which he considers may in some cases claim to be preferable to the *most advantageous method* explained in his fourth chapter."—TODHUNTER, *History of Probability*, pp. 611–612.

1818 LITTROW. 'Ueber die gerade Aufsteigung der vornehmsten Fixsterne.' *Zeitschr. f. Astron.*, Vol. VI, pp. 1–26.

The theory of Least Squares and probable errors is used.

1818 LINDENAU. 'Versuch einer Bestimmung der wahrscheinlichsten Bahn des Cometen von 1680, mit Rücksicht auf die planetarischen Störungen während der Dauer seiner Sichtbarkeit.' *Zeitschr. f. Astron.*, Vol. VI, pp. 27-120, 129–208.

The elements and their probable errors are found by the help of GAUSS's method of elimination.

1818 PLANA. 'Allgemeine Formel um nach der Methode der kleinsten Quadrate die Verbesserungen von 6 Elemente zu berechnen und zugleich das jeder derselben zukommenden Gewicht zu bestimmen.' *Zeitschr. f. Astron.*, Vol. VI, pp. 249–264.

1819 LAPLACE. 'Application du calcul des probabilités aux opérations géodésiques de la méridienne de France.' *Annal. Chem. et Phys.*, vol. XII, pp. 37–41. —*Connaiss. des Tems*, for 1822, pp. 346–348. —Trans. in *Tilloch's Phil. Mag.*, 1821, Vol. LVIII, pp. 133–136.

A discussion of the length of the meridian between Perpignan and Formentera, two points distant about 460000 metres and joined by

26 triangles. LAPLACE shows that the probable error of the computed distance is 8.194 metres. The investigation is reproduced in the Third Supplement (1820), pp. 3–7.

1819 LAPLACE. 'Mémoire sur l'application du calcul des probabilités aux observations, et specialement aux opérations du nivellement.' *Annal. Chem. et Phys.*, Vol. XII, pp. 337–341.

On the results of a discussion of the probability of errors in the computed elevation of Paris. The discussion itself is given in the Third Supplement (1820), pp. 16–28.

1819 PAUCKER. '*Ueber die Anwendung der Methode der kleinsten Quadratssumme auf physikalischen Beobachtungen.*' Mitau, 4to, pp. 32.

Published as a Gymnasium "Programm." It contains an application of the Method of Least Squares to the determination of empirical formulæ for the expansion of fluids, the specific gravity of water, and the elasticity of steam. See 1825 MUNCKE.

1819 WALBECK. *Dissertatio de forma et magnitudine telluris, ex dimensis arcubus meridiani definiendis.* Åbo, 8vo.

The first discussion of measurements of *several* arcs of meridians by the Method of Least Squares.

1819 YOUNG. Remarks on the probabilities of error in physical observations, and on the density of the earth, considered, especially with regard to the reduction of experiments on the pendulum.' *Phil. Trans. Lond.* for 1819, pp. 70–95.

Pages 70–83 are devoted to the Theory of Errors. Besides many interesting remarks it contains a method for finding probable errors, supposing that the probabilities of the several errors are proportional to the terms of the series $(1+1)^m$, m being an even number. His result for the probable error of the mean of n observations is

$$R = 0.85 \frac{\Sigma x}{n\sqrt{n}}$$

in which $\frac{\Sigma x}{n}$ denotes "the mean of all the actual errors." YOUNG refers to BESSEL, GAUSS, &c., as having used only the sum of the squares of the errors in determining the probable error, and regards his method as more accurate. See 1816.

1820 BESSEL. 'Bestimmung der geraden Aufsteigungen der 36 MASKELYNEschen Fundamental-Sterne für 1815, auf Königsberger Beobachtungen gegründet.' *Abhandl. Akad. Berlin* for 1818–19, pp. 19–36 of the mathematical part. —*Abhandl. von Bessel* (Leipzig, 1875, 4to), Vol. II, pp. 238–245.

Contains discussions on the probable errors of the observations. See also *Abhandl. Akad. Berlin* for 1825, pp. 23–35.

[1820] LAPLACE. '*Troisième Supplément à la Théorie analytique des Probabilités.*' Paris, 4to, pp. 36. —Fourth edition of the *Théorie* . . . . , pp. 624–660.

See 1819. At the end of the Supplement is an investigation of the general case of "observations assujeties à plusieurs sources d'erreurs." See TODHUNTER, *Hist. of Probability*, p. 612.

[1820] LEGENDRE. '*Nouvelles méthodes pour la détermination de l'orbite des comètes. Second supplément.*' Paris, 4to, pp. 80.

In pages 3 and 4 LEGENDRE says: "J'ai donné le premier deux méthodes sûres pour obtenir la solution à la fois la plus simple et la plus exacte, savoir: la *méthode des corrections indéterminées* . . . . . . , et la *méthode des moindres carrés* qui paraisait alors pour la première fois." See 1789 and 1805.

Pages 79–80 contain a "Note par M * * *" in which the honor of the discovery of the Method of Least Squares is claimed for LEGENDRE on the ground of priority of publication, and in which GAUSS although not mentioned by name receives several sharp hits.

1821 SVANBERG. Om roterande systemers principal-axlar och sannolikaste medel-resultatet af gifna observationer. *Vetensk. Akad. Handl. Stockholm* for 1821, pp. 388–408.

1821 ———. 'Dissertation sur la rercherche du milieu le plus probable, entre les résultats de plusieurs observations ou experiénces.' *Gergonne's Annales de Math.*, Vol. XII, pp. 181–204.

This paper discusses at some length the different methods which may be imagined for finding a mean value, and concludes that the problem is indeterminate because it is impossible to render it independent of the law of facility of error, concerning which there may be "une infinité d'hypothèses." It tries to determine a mean, first supposing that the probability of each given measurement is inversely proportional to the error committed and secondly supposing that that probability is inversely proportional to the square of the error, and concludes that the arithmetical mean can only be used when the observations differ but slightly among themselves.

The paper ends by offering a method for the correction of the arithmetical mean, which amounts to this: First find the average of the measured quantities and compute the residuals. Then take the reciprocal of each residual as the weight of its corresponding observation and find the mean of these weighted observations. Or as weights the reciprocals of the squares of the residuals may be taken. The new mean gives new residuals from which a second approximation may be made, and so on. In a note at the end, the editor (GERGONNE) suggests that this approximation will always tend to one of the given measurements as the mean.

1822 Encke. '*Die Entfernung der Sonne von der Erde aus dem Venusdurchgange von* 1761.' Gotha, 8vo, pp. 179. —Part II entitled '*Der Venusdurchgang von* 1769 . . . .' Gotha, 1824, 8vo, pp. 112.

The Method of Least Squares is applied to the reduction of 149 observations from the transit of 1761 and 106 from that of 1769, and to the determination of probable errors. The most probable distance of the sun from the earth is found to be 20666800 German geographical miles with the probable error of 89150 miles.

1822 Gauss. 'Anwendung der Wahrscheinlichkeitsrechnung auf eine Aufgabe der practischen Geometrie.' *Astronomische Nachrichten*, Vol. I, col. 81–88. —See 1855 Bertrand.

The problem is: To determine the position of a point from horizontal angles taken at that point between other points whose position is exactly known. A numerical example is given in which the number of known points is five and the number of angles is six. This is often called Pothenot's problem; see 1840 Gerling, 1866 Schott.

1823 Gauss. 'Theoria combinationis observationum erroribus minimis obnoxiæ.' *Comment. Soc. Göttingen*, Vol. V, pp. 33–90. —Also *Gauss Werke*, Vol. IV (Göttingen, 1873, 4to), pp. 1–53. —French trans. see 1855 Bertrand.

This memoir contains Gauss's second Proof of the Method of Least Squares. The following quotation from pages 37–38 shows the hypothesis upon which the proof is based: ". . . . integrale $\int xx\,\varphi(x)\,dx$ ab $x = -\infty$ usque ad $x = +\infty$ extensum (seu valor medius quadrati x^2) aptissimum videter ad incertitudinem observationum in genere definiendam et dimetiendam, ita ut e duobus observationum systematibus, quæ quoad errorum facilitatem inter se differunt, eæ præcisione prestare censeantur, in quibus integrale $\int xx\,\varphi(x)\,dx$ valorem minorem obtinet." Gauss does indeed recognize and point out that this is only an arbitrary convention, but he justifies himself in adopting it on the ground that the definition of most advantageous results must be arbitrary, since the question is in its very nature indefinite, and that his definition leads to simple operations. The values of the unknown quantities found by his method he calls "valores maxime plausibiles."

Gauss's method leads to the rule of Least Squares, whatever be the number of observations or whatever be the law of facility of error provided only that positive and negative errors are equally probable. For analyses of his proof see 1844 Ellis and 1872 Glaisher, the former regarding it as valid and the latter as unsatisfactory. In my opinion it is but little more than a begging of the question to assume that the mean of the squares of the errors is a measure of precision. See below 1825 Ivory, 1847 Galloway and 1872 Helmert.

The memoir contains an extended presentation of the practical features of the method and in this respect is of great value. The algorithm for the solution of normal equations by the method of substitution (1816), the determination of weights and of formulas for mean error occupy the second part of the memoir. The value of the mean error of an observation of the weight unity being $m=\sqrt{\frac{\Sigma x^2}{n}}$, GAUSS takes Σx^2 as referring to the true errors, and determines $m=\sqrt{\frac{\Sigma v^2}{n-q}}$ as a practical formula, Σv^2 referring to the computed residuals, n being the number of observations and q that of the unknown quantities: the investigation however is not very clear. See 1816 BESSEL and GAUSS, and 1856 BIENAYMÉ.

For GAUSS's own account of the contents of these memoirs see the *Göttingische gelehrte Anzeigen*, Feb. 26, 1821 and Feb. 24, 1823. These reviews are reprinted in Vol. IV. of *Gauss Werke*, pp. 95–104. GAUSS here states that in the year 1797 he found that the determination of the *most probable* values of observed quantities was impossible, unless the law of facility of error was known; and that since 1801 he had used the Method of Least Squares almost daily. See 1830 RIESE.

1824 BERLIN. *Explanatio methodi quadratorum minimorum.* Lundæ, 4to.

1824 FOURIER. 'Règle usuelle pour la recherche des résultats moyens d'un grand nombre d'observations.' *Férussac's Bull. Sci. Math.*, Vol. II, pp. 88–90.

The rule given is expressed by the formula

$$m=0\cdot4769\,\frac{\sqrt{\frac{\Sigma a^2}{n}-\left(\frac{\Sigma a}{n}\right)^2}}{\sqrt{2n}}$$

$a_1, a_2, a_3, \ldots$ being the results of the n observations.

1824 POISSON. 'Sur la probabilité des résultats moyens des observations.' *Connaiss. des Tems* for 1827, pp. 273–302; for 1832, pp. 3–22.

These memoirs are a commentary on LAPLACE's fourth Chapter (1812) and seem to form a kind of translation which POISSON made of LAPLACE's investigations for his own satisfaction. A large part of the memoirs are reproduced in his *Recherches* . . ., see 1837. See also 1830 HAUBER, 1847 GALLOWAY and TODHUNTER's *History of Probability*, pp. 560–588. See *Jahrb. Chem. u. Phys.*, Vol. IV, pp. 38–42.

"POISSON confines himself to the case in which *one* element is to be determined from a large number of observations, but he treats this case in a more general manner than LAPLACE had done. LAPLACE had assumed that positive and negative errors were equally likely, and that the law of facility of error is the same at every observation; but POISSON makes neither of these assumptions."—TODHUNTER in *Trans. Camb. Phil. Soc.*, 1869, Vol. XI, p. 219.

1824 PUISSANT. 'Application de la methode la plus avantageuse à la determination de l'aplatissement du sphéroïde osculateur en France, par le comparaison d'un arc de méridien avec un arc de parallèle.' *Férussac's Bull. Sci. Math.*, Vol. I, pp. 271–274. —*Connaiss. des Tems* for 1827, pp. 230–232.

A general statement of the method, which seems to have been elsewhere published in detail.

1825 IVORY. 'On the Method of the Least Squares.' *Tilloch's Phil. Mag.*, Vol. LXV, pp. 3–10, 81–86, 161–168.

This paper contains two attempted proofs of the principle of Least Squares by methods independent of the Theory of Probability. The first, in page 5, rests on a vague analogy with the properties of a lever and is in the words ef ELLIS "little more than a *petitio principii* concealed by a metaphor." The second, in pages 6–7, rests on the supposition that "the mean of the sum of the squares of the errors may be taken as a measure of the precision of the observations" which can scarcely be assumed as evident; this is similar to GAUSS's proof of 1823.

Pages 81–86 are devoted to discussing the probability of errors. IVORY makes no distinction between true errors and residuals, and does not recognize that the probability of any definite error must be an infinitessimal. The remaining pages attempt to show that the Method of Least Squares cannot give the most advantageous or probable results unless the law of facility of error is $\varphi(x) = ce^{-h^2x^2}$, and that LAPLACE's demonstration "whatever merit it may have in other respects is neither more or less general than the other solutions of the problem."

These two proofs are examined and exposed by ELLIS in 1844, and the second proof with the criticisms on LAPLACE are analyzed by GLAISHER in 1872. See also 1851 HOSSARD.

1825 MUNCKE. 'Beobachtung.' Art. in *Gehler's Physikalisches Wörterbuch*, second ed. Vol. I, pp. 884–912.

Contains matter from 1823 GAUSS and 1819 PAUCKER.

1826 GAUSS. 'Chronometrische Längenbestimmung.' *Astron. Nachr.*, Vol. V, col. 227–240, 245–248. —See also 1855 BERTRAND.

The Method of Least Squares is used and formulæ for finding mean errors of the results given.

1826 [FOURIER]. Recherches statistiques sur la ville de Paris et le département de la Seine [par le Compte CHABROL-DE-VOLVIC]. Paris, 4to, Vol. III, in 1826, Vol. IV in 1829.

These volumes contain investigations by FOURIER on weights, probable errors, etc. See note by DEMORGAN in *Lond. Assur. Mag.*, Vol. XIV, p. 89.

1826 IVORY. 'On the Method of the Least Squares.' *Tilloch's Phil. Mag.*, Vol. LXVIII, pp. 161-165.

Contains IVORY's third attempted proof of the Method, which is still more absurd than those of 1825. See ELLIS's analysis in his paper of 1844. See also below, 1830 FRANCOEUR.

1827 ROSENBERGER. 'Ueber die, auf Veranstaltung der französischen Academie, während der Jahre 1736 und 1737 im Schweden vorgenommene Gradmessung.' *Astron. Nachr.*, Vol. VI, col. 1-32.

An application of a method, communicated to the author by BESSEL, for the adjustment of geodetic triangulations by the use of Least Squares. See 1831 HANSEN. See *Abhandl. von Bessel* (Leipzig, 1875), Vol. III, pp. 16-19.

1827 ———. Ueber die Theorie der Zuverlässigkeit der Beobachtungen und Versuche und der von derselben abhängigen Bestimmungen des Mittels aus gegebenen Zahlen. [Berlin].

1828 BESSEL. 'Ueber die Bestimmung des Gesetzes einer periodischen Erscheinung.' *Astron. Nachr.*, Vol. VI, col. 333-348. —*Abhandl. von Bessel* (Leipzig, 1875, 4to), Vol. II, pp. 364-372.

Proposes the periodic function since so much used for discussing recurring phenomena, and illustrates its application to the determination of empirical formulæ. See 1864 SCHOTT.

1828 GAUSS. 'Supplementum theoriæ combinationis observationum erroribus minimis obnoxiæ.' *Comment. Soc. Göttingen*, Vol. VI, pp. 57-98. —Also *Gauss Werke*, Vol. IV, pp. 57-93. —French trans. see 1855 BERTRAND.

This memoir discusses a method for the combination of observations when the observed quantities are not expressed as explicit functions of the unknown quantities to be determined; and when the problem furnishes rigorous equations of condition which the determined values of the unknown quantities must exactly satisfy. The *method of correlatives* for the adjustment of such conditioned observations is given and an algorithm presented for its use. A numerical example involving twenty-four observations subject to thirteen conditions illustrates the use of the formulæ. See the *Gött. gelehrte Anzeigen*, Sept. 25, 1826, or *Gauss Werke*, Vol. IV, pp. 104–108 for GAUSS's own account of the contents of the memoir. See also 1830 RIESE.

1828 QUETELET. *Instructions populaires sur le calcul des probabilités*, Bruxelles, 18mo.

1830 FRANCOEUR. 'De la composition des formules astronomiques, et de la détermination des constantes; Équations de condition; Méthodes de TOBIE MAYER et des moindres carrés.' *Astronomie pratique* (Paris, 8vo; second ed., 1840), pp. 412–431.

IVORY's proof of 1826 is given as a perfectly valid "démonstration." This is repeated in the second edition.

1830 HANSEN. Commentatio de gradus præcisionis computatione. [Gotha, 4to.]

This was first printed "in einem Program, womit die hiesige Sternwarte [at Gotha] des Jubiläum des würdigen OLBERS gefeiert hat." It contained a method for finding the weights of values determined by the Method of Least Squares, which for a small number of unknown quantities is perhaps shorter than that of GAUSS (1823). See *Astron. Nachr.*, Vol. VIII, col. 462–463, and ENCKE in *Berlin Astron. Jahrb.* for 1835, p. 297.

1830 HAUBER. Ueber die Bestimmung der Genauigkeit der Beobachtungen. *Baumgartner's Zeitschr. f. Phys. Math., etc.*, Vol. VII, pp. 286–314.

1830 HAUBER. Verallgemeinerung der POISSON'schen Untersuchungen über die Wahrscheinlichkeit der mittlern Resultate der Beobachtungen in den "Additions à la *Connaissance des Temps* de 1827." *Baumgartner's Zeitschr.*, Vol. VII, pp. 406–429.

1830 HAUBER. Theorie der mittleren Werthe. *Baumgartner's Zeitschr.*, Vol. VIII, pp. 25–26, 147–179, 295–315, 443–445; Vol. IX. pp. 302–322; Vol. X, pp. 425–457.

[1830] NÜRNBERGER. Betrachtung über die Methode der kleinsten Quadrate.

1830 POISSON. 'Note sur la probabilité du résultat moyen des observations.' *Férussac's Bull. Sci. Math.*, Vol. XIII, pp. 266–277.

On LAPLACE's method of deducing formulæ for probable errors.

1830 POISSON. 'Mémoire sur la proportion des naissances des filles et des garçons.' *Mém. Acad. Paris*, Vol. IX, pp. 239–308.

The general term of the binomial theorem is shown to take the exponential form ce^{-u^2}, when the number of terms is indefinitely great: See LAPLACE, *Théorie des .... Prob.*, Chap. III, and 1837 HAGEN. Formulæ for probability of errors between given limits and for probable errors are also developed. See 1836 POISSON.

1830 RIESE. '[Eine Recension.]' *Jahrb. für wissen. Kritik* for 1830, Pt. I, col. 269–284.

A review of Gauss's memoirs *Theoria combinationis .....*, 1823, and *Supplementum theoriae. ....*, 1827, giving an account of their contents and a popular exposition of the subject.

1831 CAUCHY. 'Mémoire sur le système de valeurs qu'il faut attribuer à divers Élémens, déterminés par un grand nombre d'observations, pour que la plus grande de toutes les erreurs, abstraction faite du signe, devienne un minimum.' *Jour. École Poly.*, Vol. XIII (cahier 20), pp. 175–221.

This was perhaps published about 1814, See *Bull. Soc. Philom. Paris*, for 1824, pp. 92–99. The methods of 1760 BOSCOVICH and 1792 LAPLACE, are particular cases of CAUCHY's solution. See LAPLACE's *Théorie ... des Prob.*, Chap. IV, Art. 24.

1831 DEGEN. 'Recherches sur la parabole, déterminée par la méthode des moindres carrés et qui représenté le moins défectueusement, quil il soit possible, un système quelconque de points donnés dans un plan.' *Mém. Acad. St. Péters. par divers savants*, Vol. I, pp. 13–28.

1831 GRUNERT. 'Berechnung der wahrscheinlichsten Resultate aus gegebenen Beobachtungen. Methode der kleinsten Quadrate.' *Klügel's Mathematisches Wörterbuch*, Vol. V, Art. "Wahrscheinlichkeitsrechnung,' pp. 983–1027.

A clear presentation of the Method of Least Squares according to GAUSS, the proof being that of 1809.

1831 HANSEN. 'Darlegung einer neuen Methode, bei Anwendung der Methode der kleinsten Quadrate, die Gewichte der unbekannten Grössen zu berechnen.' *Astron. Nachr.*, Vol. VIII, col. 463–468.

A simplification of GAUSS's method of 1823 for finding weights. Not the same as 1830 HANSEN. See 1832 ENCKE.

1831 HANSEN. 'Ueber die Anwendung der Wahrscheinlichkeitsrechnung auf geodätische Vermessungen im Allgemeinen, und über die MAUPERTUIS'sche Gradmessung.' *Astron. Nachr.*, Vol. IX, col. 189–220, 237–262.

The measurements are discussed by GAUSS's method of correlatives (1828) whose algorithm is given in full, and also by a new method of HANSEN. ROSENBERGER had in 1827 examined the same measurements.

1831 LITTROW. Bemerkungen zum practischen Gebrauche der Wahrscheinlichkeitsrechnung. *Baumgartner's Zeitschr. f. Phys.*, Vol. IX, pp. 433–449.

1831 PUISSANT. 'Application du calcul des probabilités à la mesure de la precision d'un grand nivellement trigonométrique.' *Mém. Acad. Paris*, Vol. X, pp. 533–547. —*Connaiss. des Temps* for 1834, pp. 3–17.

A modification of the method of LAPLACE's Third Supplement (1820), illustrated by a practical example.

1832 ENCKE. 'Ueber die Begründung der Methode der kleinsten Quadrate.' *Abhandl. Akad. Berlin* for 1831, pp. 73–78 of the mathematical part.

After brief notices of five proofs of the Method of Least Squares, ENCKE gives the preference to GAUSS's of 1809. To establish this more rigidly he offers a demonstration to show that for direct observations the rule of the arithmetical mean gives the most probable result. This demonstration (in my opinion not a rigorous one) has been followed by many subsequent writers. It is repeated by ENCKE in the article quoted next below, and is particularly stated with confidence by CHAUVENET in 1864. For criticisms see 1843 REUSCHLE and 1872 GLAISHER. See also ENCKE's later opinion, below under 1850.

1832 ENCKE. 'Ueber die Methode der kleinsten Quadrate.' *Berlin. Astron. Jahrbuch* for 1834, pp. 249–312; for 1835, pp. 253–320; for 1836, pp. 253–308. —Republished in *Encke's astronomische Abhandlungen* (Berlin, 1866), Vol. I, Nos. xii, xiii, xiv.

These memoirs form a treatise on the Method of Least Squares, from which many text-books have been compiled.

The first memoir contains the proof of 1809 GAUSS, reinforced by ENCKE's attempted demonstration of the validity of the arithmetical mean, the discussion of weight and probable errors, and two tables of the probability integral $\frac{2}{\sqrt{\pi}}\int e^{-t^2}dt$, the first between the limits 0 and t, and the second between the limits 0 and $0.476936\frac{x}{r}$ (x being any error and r the probable error). These were computed from KRAMP's tables of 1799 as quoted by BESSEL in 1818. See *Lond. Phil. Mag.*, 1871, Vol. XLII, p. 431, et sq. A translation of this first memoir and a reprint of the tables is given in *Taylor's Scientific Memoirs*, 1841, Vol. II, pp. 317–369.

ENCKE takes $y=\varphi(x)$ as the equation of the curve expressing the probability of error, and regarding x and y as continuous variables recognizes clearly that for a given error $\varphi(x)$ must be an infinitesimal. But strange to say he deduces

$$\varphi(x)=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$$

in which h is a finite quantity.

The second memoir contains the practical features of the method—GAUSS's algorithm for the solution of normal equations, GAUSS's (1823) and HANSEN's (1830 and 1831) methods of determining weights, etc. The third is devoted to the discussion of conditioned observations. At the time of publication these memoirs must have been of great value to students.

1832 LITTROW. *Die Wahrscheinlichkeitsrechnung und ihre Anwendung auf das wissenschaftliche und practische Leben.* Wien, 8vo.

1832 PUISSANT. 'Deuxième mémoire sur l'application du calcul des probabilités aux mesures géodésiques.' *Mém. Acad. Paris*, Vol. XI, pp. 123–156.

Adjustment of triangulations, determinations of probable errors, etc.

1834 BESSEL. 'Betrachtung über die Methode der Vervielfältigung der Beobachtungen.' *Astron. Nachr.*, Vol. XI, col. 269–290. —*Abhandl. von Bessel* (Leipzig, 1875), Vol. III, pp. 306–317.

This valuable paper deduces rules for the adjustment of angles taken by the method of repetitions, and formulæ for finding their weights and probable errors.

1834 STROOTMAN. *Bevattelijk onderrigt in de Kansrekening, of de leer der waarschijnlijkheden.* Breda, 12mo.

1835 CAUCHY. 'Mémoire sur l'interpolation.' Lith. MS. —Translation in *Lond. Phil. Mag.*, 1836, Vol. VIII, pp. 459–468. —Reprinted in *Liouville's Jour. Math.*, 1837, Vol. II, pp. 193–205; in *Moigno's Leçons de calcul différentiel* (Paris, 1840), pp. 513–526.

When an empirical formula is to be derived from a great number of observation equations, CAUCHY's method may be used as easily, although perhaps with less accuracy than the Method of Least Squares. See below 1853 BIENAYMÉ and CAUCHY, 1842 GRUNERT and 1861 SCHOTT. See an article by BARTLETT in *Amer. Jour. Sci.*, 1862, Vol. XXXIV, pp. 27–33.

1836 POISSON. 'Formules relatives aux probabilités qui dépendent de très grand nombres.' *Comptes Rend. Acad. Paris*, Vol. II, pp. 603–613.

The general term of the binomial $(p+q)^m$, in which $p+q=1$, is shown to approach the form $\frac{e^{-u^2}}{\sqrt{2\pi mpq}}$, as m indefinitely increases. See 1830 POISSON.

1836 ROUVROY. Ueber die Methode der kleinsten Quadrate. Appendix to his *Mechanik* (Dresden and Leipzig, 8vo).

1837 HAGEN. *Grundzüge der Wahrscheinlichkeits-Rechnung.* Berlin, 8vo. —Second edition, see 1867.

This work contains HAGEN'S proof of the Method of Least Squares. It is based upon the following hypothesis: "Der Beobachtungsfehler ist die algebraische Summe einer unendlich grossen Anzahl elementärer Fehler, die alle gleichen Werth haben und eben so leicht positiv, wie negativ sein können." This postulated, the proof consists in finding the general term of the expansion of $(\frac{1}{2}+\frac{1}{2})^{2m}$, m being indefinitely large. The law of facility of error takes the form $\varphi(x)=(\pi m)^{-\frac{1}{2}}e^{-\frac{x^2}{m}}$ from which the principle of Least Squares at once follows.

The algebraic work of HAGEN'S method had in a somewhat different form been given by LAPLACE in the *Théorie .... des Probabilités* (1812), p. 301, and in the articles 1830 POISSON and 1836 POISSON. HAGEN'S method is more elementary and in connection with his original hypothesis forms, I think, one of the best proofs of the Method of Least Squares.

HAGEN'S proof is given in the writings 1849 WITTSTEIN, 1850 ENCKE, 1852 DIENGER, and in a modified form 1846 QUETELET, 1865 TAIT, 1866 NATANI and others. Also see PRICE'S *Integral Calculus* (Oxford, 1865), pp. 376–379. A discussion between KUMMELL and MERRIMAN concerning this proof is now (Oct., 1877) going on in the *Jour. Franklin Institute;* see Vol. CIV, pp. 173–187, 270–274, et sq.

1837 POISSON. '*Recherches sur la probabilité des jugemens en matière criminelle et en matière civile, précédées des règles générales du calcul des probabilités.*' Paris, 4to, pp. ix, 415. —German trans. by SCHNUSE called *Lehrbuch der Wahrscheinlichkeitsrechnung;* Braunschweig, 1841, 8vo.

The matter of POISSON'S previous memoirs on the law of great numbers is reproduced in Chap. III, and of those on the probability of the mean in Chap. IV.

1838 BESSEL and BAEYER. '*Gradmessung in Ost-Preussen und ihre Verbindung mit preussische und russische Dreieckskettten.*' Berlin, 4to, pp. xiv, 452.

A geodetic work of great value, containing many applications of the Method of Least Squares. See extracts in *Abhandl. von Bessel,* (Leipzig, 1875), Vol. III, pp. 82–138.

1838 BESSEL. 'Untersuchungen über die Wahrscheinlichkeit der Beobachtungsfehler.' *Astron. Nachr.*, Vol. XV, col. 369–404. —*Abhandl. von Bessel* (Leipzig, 1875), Vol. II, pp. 372–391.

"Ich werde nämlich die Entstehungsart der Beobachtungsfehler *aus ihren Ursachen*, zum Grunde des Folgenden machen. Wenn man aufangs die Fehler einer gewissen Beobachtungsart also aus *einer*, auf gegebene Art wirkenden Ursache hervorgehend betrachtet, so wird dadurch ihre jedesmalige Grösse x eine gegebene Function eines Arguments ξ, welches in derselben Art willkührlich ist, wie das Fallen eines Würfels. Aus dem Ausdrucke $x = f\xi$ kann aber der Ausdruck $\varphi(x)$ abgeleitet werden, ..."

BESSEL seems to use the word *Ursache* in the sense of *a source of error.* His first investigation is of a case arising in the measurement of angles, where the error x is related to the *Ursache* ξ by the law $x = a \sin \xi$, every value for ξ between the limits $\pm \frac{1}{2}\pi$ being equally possible. The law of facility of error he finds $\varphi(x) = \pi^{-1}(a^2 - x^2)^{-\frac{1}{2}}$, and the probable error is 2.568 times greater than by the Method of Least Squares. An example where $x = a\xi^2$, which he shows may actually arise, gives also disagreeing results.

In the second part of the investigation we read: "Ich werde nun die Wahrscheinlichkeit eines Fehlers untersuchen, welcher aus der *Zusammenwirkung* mehrer, von einander unabhängiger Ursachen ensteht," each error so arising being considered as equally likely to be positive or negative. The result of the investigation is that the law of facility of error approximates closely to the exponential form $\varphi(x) = ce^{-h^2x^2}$, provided that "viele Ursachen zur Hervorbringen des Beobachtungsfehlers *zusammenwirken*," and "dass unter den, aus den einzelen Ursachen hervorgehenden mittlern Fehlern, keiner die ubrigen beträchtlich übertreffe," and these conditions BESSEL thinks, are present in most observations.

This memoir is very valuable as showing that the exponential law of facility is not to be regarded as an *à priori* rule, free from exception, and as throwing new light on the condition under which it exists. On the whole it may be considered as a new proof of that law and hence of the Method of Least Squares.

1838 BIENAYMÉ. 'Mémoire sur la probabilité des résultats moyens des observations; démonstration directe de la règle de LAPLACE.' *Mém. . . par divers savants*, . . *Acad. Paris*, Vol. V, pp. 513–558.

The rule of LAPLACE here meant is a method for finding the probability of the error of the mean. The opening pages contain some interesting historical remarks, but the investigation itself is very long and tedious and seems to be of little value.

1838 DEMORGAN. '*Essay on Probabilities*,' London, 12mo. pp. xviii, 306, XL.

This popular book devotes a chapter to methods of finding weights and estimating probabilities of mean results. It contains tables of the error functions.

1838 DEMORGAN. 'On a Question in the Theory of Probabilities.' *Trans. Camb. Phil. Soc.*, Vol. VI, pp. 423–430.

"The object of this paper is the correction of an oversight made both by LAPLACE and M. POISSON in pages 279 and 209 of their respective works on the Theory of Probabilities."

1838 PUISSANT. 'Sur l'application du calcul des probabilités à la mesure de la precision d'un grand nivellement géodésique independant des distances respectives des stations.' *Comptes Rend. Acad. Paris*, Vol. VII, pp. 1132–1135.

The mean errors of the adjusted elevations are deduced.

1839 BIENAYMÉ. Théorème sur la probabilité des résultats moyens des observations. *Procès-Verb. Soc. Philom. Paris* for 1839, pp. 60–65.

1839 HANSEN. 'Auflösung einer allgemeinen Aufgabe aus der Wahrscheinlichkeitsrechnung.' *Astron. Nachr.*, Vol. XVI, col. 9–16, 27–32.

On the determination of the values and weights of observed quantities subject to conditional equations. A particular case of HANSEN's problem is noted below under 1841.

1839 JAHN. *Die Wahrscheinlichkeitsrechnung und ihre Anwendung auf das wissenschaftliche und praktische Leben.* Leipzig, 8vo, pp 240.

1840 BESSEL. 'Neue Formel von JACOBI, für einen Fall der Anwendung der Methode der kleinsten Quadrate.' *Astron. Nachr.*, Vol. XVII, col. 305–308. —*Abhandl. von Bessel* (Leipzig, 1875), Vol. II. pp. 401–402.

Formulæ for solving normal equations containing *three* unknown quantities and determining at the same time the weights. They are regarded by BESSEL as shortening considerably the numerical work required by previous methods. See below 1873 SEELIGER.

1840 BESSEL. 'Ein Hülfsmittel zur Erleichterung der Anwendung der Methode der kleinsten Quadrate.' *Astron. Nachr.*, Vol. XVII, col. 225–230. —*Abhandl. von Bessel*, Vol. II, pp. 398–401.

Computations arising in the formation of normal equations can be shortened by the use of tables of squares of numbers.

1840 BIENAYMÉ. La constance des résultats moyens. *Procès-Verb. Soc. Philom. Paris* for 1840, pp. 19–26.

1840 BIENAYMÉ. Principe des probabilités entièrement nouveau. *Procès-Verb. Soc. Philom. Paris* for 1840, pp. 37–43.

1840 BIENAYMÉ. Quelques proprietés des moyens arithmétiques de puissances de quantities positives. *Procès-Verb. Soc. Philom.* for 1840, pp. 67–70.

1840 BRAVI. *Theorica e practica del probabile.* Bergamo, second edition, 2 vols, 8vo.

1840 CLEMENS. 'Ueber die Methode der kleinsten Quadrate. *Tilsit Gymnasium Programm* for 1839–40, pp. ....; for 1848, pp. 1–12.

The first part of this memoir I have not seen. The second discusses the determination of empirical formulæ for periodic observations, the calculation of probable errors and the correction of a field survey.

1840 GERLING. POTHENOT'*sche Aufgabe, in practischer Beziehung dargestellt.* Marburg, 8vo, pp. 32.

Contains mainly the solution of 1822 GAUSS. See 1866 SCHOTT.

1841 BESSEL. 'Ueber einer Fehler in der Berechnung der französischen Gradmessung und seinen Einfluss auf die Bestimmung der Figur der Erde.' *Astron. Nachr.*, Vol. XIX, col. 97–116. —*Abhandl. von Bessel* (Leipzig, 1875), Vol. III, pp. 55–62.

Forty-four observations subject to sixteen conditions are adjusted. The method here employed for the combination of measurements of arcs of meridians has furnished a model for many subsequent investigations.

1841 HANSEN. 'Eine Aufgabe aus der practischen Geodäsie und deren Auflösung.' *Astron. Nachr.*, Vol. XVIII, col. 165–176.

"Die Lage *zweier unbekannten* Punckte durch Hülfe der Lage *zweier bekannten* Punckte zu bestimmen, ohne jene von diesen aus zu beobachten."

1841 HÜLSSE. *Ueber die Berechnung von Beobachtungen durch die Methode der kleinsten Quadratssumme.* Leipzig, 4to.

1841 JACOBI. 'De formatione et proprietatibus Determinantium.' *Crelle's Jour. Math.*, Vol. XXII, pp. 285–318.

Near the end of this memoir are remarks concerning the application of determinants to the solution of equations and to the finding of weights in the Method of Least Squares.

1841 QUETELET. Ueber das Gesetz des Quadrats der Temperaturen. *Bull. Akad. München* for 1841, col. 261–263.

1842 FRIES. *Versuch einer Kritik der Principien der Wahrscheinlichkeitsrechnung.* Braunschweig, 8vo.

1842 GRUNERT. 'Ueber CAUCHY's Interpolationsmethode.' *Archiv. Math. u. Phys.*, Vol. II, pp. 41–60.
See 1835 CAUCHY.

1842 LITTROW. 'Theorie der kleinsten Quadrate.' *Gehler's Physicalisches Wörterbuch*, Vol. X, Pt. II, Art. 'Wahrscheinlichkeitsrechnung,' pp. 1200–1251.
A clear elementary exposition of the Method and its theory according to 1832 ENCKE.

1842 LOBATSCHEWSKY. 'Probabilité des résultats moyens tirés d'observations répetées.' *Crelle's Jour. Math.*, Vol. XXIV, pp. 164–170.
Serial expressions are deduced for the probability of error in the mean of a limited number of observations.

1842 MERZ. *De theoria probabilitatis adhibita in physicam.* Monachii.

1842 PUISSANT. '*Traité de géodésie, ou exposition des méthodes trigonométriques et astronomiques, applicables à la mesure de la terre, et à la construction du canevas des cartes topographiques.*' Paris, 2 vols, 4to, Vol. I, pp. xvi, 515, and XI pl. Vol. II, pp. xi, 496, xxxii, and III pl. —First edition, see 1805.
The results of many of PUISSANT's previous memoirs are here recorded, and the Method of Least Squares is often used. The work is one of the most valuable treatises on Geodesy extant.

1842 RAMUS. 'Sur une question de probabilité relative aux corrections des hauteurs barométriques.' *Crelle's Jour. Math.*, Vol. XXIV, pp. 80–84.
A determination of the probability "que dans un grand nombre d'observations la difference des résultats moyens des hauteurs barométriques observées et réduites soit comprise dans l'intrevalle λ."

1842 RÖBER. Experiment. Art. in *Handwörterbuch Chemie u. Phys.* (Berlin, 8vo.)
See *Archiv. Math. u. Phys.*, Vol. XI, p. 375.

1843 ARGELANDER. 'Ueber die Anwendung der Methode der kleinsten Quadrate auf einen besondern Fall.' *Astron. Nachr.*, Vol. XXI, col. 163–168.

Simplifies the formation of normal equations, etc., when the observations are made on the simple sums or differences of the unknown quantities.

1843 BERKHAN. *Ueber die Methode der kleinsten Quadrate.* Blankenburg, 8vo.

1843 COURNOT. '*Exposition de la théorie des chances et des probabilités.*' Paris, 8vo, pp. viii, 448. —German trans., Braunschweig, 1849, 8vo.

Chapters on the theory of means and the adjustment of observations are given. At the end is a table of values of the error function. COURNOT'S methods are often quite awkward.

1843 GERLING. '*Die Ausgleichungs-Rechnung der practischen Geometrie, oder die Methode der kleinsten Quadrate mit ihren Anwendungen auf geodätische Aufgaben.*' Hamburg and Gotha, 8vo, pp. xix, 409, and 4 pl.

This book is dedicated to GAUSS. The principle of Least Squares is assumed as the basis of the methods of adjustment. It consists of four parts; the first treats of direct, the second of indirect, and the third of conditioned observations, while the fourth discusses the form and number of conditional equations which need to be considered in the adjustment of triangulations.

The book is fully illustrated with practical examples, and contains the best systematized development of the application of the method to the treatment of simple geodetic measurements that has yet appeared. GERLING issued later (1845, 1855, 1862) some papers supplementary to the work, one of which contains a long list of errata.

1843 REUSCHLE. 'Ueber die Deduction der Methode der kleinsten Quadrate aus Begriffen der Wahrscheinlichkeitsrechnung.' *Crelle's Jour. Math.*, Vol. XXVI, pp. 333–364; Vol. XXVII, pp. 182–184.

This is a critical examination of the deduction of the expression for the probability of an error between given limits and of other points in the "Metaphysik" of the Method. ENCKE'S proof of the validity of the arithmetical mean (1832) is also examined and found to be imperfect.

1844. ENCKE. 'Bemerkungen zu der Abhandlung No. 22. Band 26. Heft 4, dieses Journals.' *Crelle's Jour. Math.*, Vol. XXVIII, pp. 213–222.

A reply to REUSCHLE's criticisms. ENCKE tries to explain that in the two expressions

$$\Sigma_0^a \varphi(x) = 1 \quad \text{and} \quad \int_0^a \varphi(x)\, dx = 1$$

the symbol 1 has different meanings. He also shows that REUSCHLE had failed to understand his proof of the arithmetical mean.

1844 DONKIN. An Essay on the Theory of the Combination of Observations. *Trans. Ashmolean Soc.* (Oxford. 8vo). —French abridgment, 'Sur la théorie....' in *Liouville's Jour. Math.* 1855, Vol. XV, pp. 297–322.

DONKIN attempts to establish "une espèce de Statique métaphysique sur des preuves de la même force que celles qu'on emploie en déduisant, à priori, les lois de la Statique ordinare." The word "force" is taken to mean "tout motif qui nous porte à altérer la valeur attribuée à une quantité," and to these "forces" the principles of centre of gravity of bodies, of virtual velocities, etc., are applied, and the usual rules for the adjustment of observations by means of normal equations, weights, mean errors, etc., are deduced. No law of facility of error enters into the discussion.

DONKIN's reasoning does not always seem to me clear or rigorous.

1844 ELLIS. 'On the Method of Least Squares.' *Trans. Camb. Phil. Soc.*, Vol. VIII, pp. 204-219. —Also *Ellis's Mathematical and other Writings* (Cambridge, 1863, 8vo), pp. 12–37.

In this paper it is attempted "to bring the different modes in which the subject has been presented into juxtaposition, as that the relations which they bear to one another may be clearly apprehended."

ELLIS first takes up GAUSS's proof of 1809. He considers that GAUSS is not justified in assuming that the rule of the arithmetical mean gives the most probable values, and he shows that besides mere convenience no satisfactory reason can be assigned why it should be so regarded. His remarks on this point are extremely valuable and sound. See 1872 GLAISHER.

LAPLACE's demonstration is taken up and presented in a different but greatly simplified form, extended to the case of any number of unknown elements. GAUSS's second proof of 1823 is also analyzed and the conclusion arrived at that "nothing can be simpler or more satisfactory." Lastly IVORY's three proofs (1825-6) are discussed and their illogical character clearly exposed. The paper is one of the most valuable in the theoretical literature of the subject.

1844 JACOBI. 'Ueber eine neue Auflösungsart der bei der Methode der kleinsten Quadrate vorkommenden lineären Gleichungen.' *Astron. Nachr.*, Vol. XXII, col. 297–306.

An abridged method for the solution of certain forms of normal equations.

[1844] QUETELET. Sur l'appréciations des documents statistiques et en particular sur l'appréciations des moyennes. *Bull. statistique de Belgique*, Vol. II.

1845 BESSEL. 'Auszug aus einem Schreiben....' *Astron. Nachr.*, Vol. XXIII, col. 1–4.

On the probable errors of interpolations in logarithmic tables.

1845 FISCHER. Die Theorie der Beobachtungsfehler und ihre Ausgleichung durch die Methode der kleinsten Quadrate. Pt. I of his *Lehrbuch der höheren Geodäsie;* Darmstadt, 8vo.

1845 GERLING. 'Nachträge zur Ausgleichungs-Rechnung.' *Archiv. Math. u. Phys.*, Vol. VI, pp. 141–146, 375–378.

Three supplements to GERLING'S book (1843). The first and second treat of the determination of points by angle measurements and the third of the precision of chain measurements.

1846 BRAVAIS. 'Analyse mathématique sur les probabilités des erreurs de situation d'un point.' *Mém. . . par divers savans . . Inst. France*, Vol. IX, pp. 255–332.

Observations being made on the coördinates of a point the probability that the apparent and true places are separated by a given distance is investigated, as also is the "valeur de la crainte mathématique de l'erreur" which is shown to be represented by the surface of a certain ellipsoid. The method requires the observations to be numerous.

In determining the probability that a point in a plane is located on an elementary area $dy.dx$, BRAVAIS takes the probabilities of the x and y deflections as independent. See 1850 HERSCHEL.

1846 GALLOWAY. 'On the application of the Method of Least Squares to the Determination of the most probable Errors in a Portion of the Ordnance Survey of England.' *Mem. Astron. Soc. Lond.*, Vol. XV, pp. 23–69.

A discussion of a triangulation including ten stations at which thirty-five angles are observed subject to nineteen conditional equations. The methods of adjustment, of determining weights, of forming the conditional equations and all the steps of the process are given at length with great clearness. See *Month. Notices Astron. Soc.*, 1843, Vol. V, pp. 262–264 for a full account of contents of the memoir.

1846 QUETELET. '*Lettres à S. A. R. le Duc régnant de Saxe-Cobourg et Gotha, sur la Théorie des Probabilités appliquée aux sciences morales et politiques.*' Bruxelles, large 8vo, pp. iv, 450. —English trans. by DOWNES; London, 1849, 8vo.

These letters are of so elementary a character that S. A. R. is informed of the meaning of the signs + and —. They contain however a valuable popular exposition of the Theory of means and of the laws of error. For review of the book see 1850 HERSCHEL.

At the end of the book is an appendix containing many valuable "Notes." In pages 375–380 a table of the terms of the binomial $(\frac{1}{2}+\frac{1}{2})^{999}$ for eighty terms on each side of the middle term is given and the method of its computation explained: these have since been called QUETELET's numbers, see 1869 GALTON. In pages 384–387 it is shown that the general term of the binomial $(\frac{1}{2}+\frac{1}{2})^m$ approaches the exponential form $ce^{-h^2x^2}$ as m indefinitely increases: this is similar to HAGEN's investigation of 1837.

In pages 412–424 are printed three suggestive letters from BRAVAIS, in which not only doubts are expressed as to the *à priori* necessity of the exponential law of facility of error, but examples are given to show that it is not universally true *à posteriori.* BRAVAIS' view is that every cause of partial error gives rise to a distinct curve of facility and that the combination of these approaches the exponential form as a limit, partly because of the necessary law that positive and negative errors are equally likely, and partly because the combination itself must tend toward the binomial form. He alludes to HAGEN's proof as not sufficiently rigorous.

1847 DEMORGAN. 'Theory of Probabilities.' *Encycl. of Pure Math.* (*Encycl. Metrop.*), Pt. II, pp. 393–490.

A great part of this work is translated and adapted from LAPLACE's *Théorie .... des Prob.*, 1812, enriched by comments. The Method of Least Squares is treated at considerable length according to LAPLACE's method. At the end are given valuable tables, those of 1799 KRAMP and 1832 ENCKE, and also factorial tables.

1847 GALLOWAY. 'Probabilities.' *Encycl. Brittanica*, seventh ed., Vol. XVIII, pp. 591–639. Eighth edition, Vol. XVIII, pp. 588–636. —Also separately, Edinburgh, 1848, 8vo.

POISSON's analysis (1824) of LAPLACE's method is given, and also GAUSS's proof of 1823.

1848 MATZKA. 'Beweis des obersten Grundsatzes der Methode der kleinsten Quadrate.' *Archiv. Math. u. Phys.*, Vol. XI, pp. 369–377.

A suggestive article. Let x be the true value of a quantity for which observations give the values $a, b, c, \ldots$, then

$$x=f(a, b, c, \ldots)$$

and also we must have

$$x-m=f(a-m, b-m, c-m, \ldots).$$

Applying TAYLOR's theorem, MATZKA deduces for x

$$x=\frac{ha+ib+kc+\ldots}{h+i+k+\ldots}$$

in which, h, i, k, ... are positive quantities whose values cannot be determined. If the observations are of equal weight and infinite in number the rule of the arithmetical mean is shown to follow.

1848 MEYER. Résolution d'un problême du calcul des probabilités. *Bull. Acad. Belgique,* Vol. XV, Pt. II, pp. 508–512.

1848 MEYER. 'Mémoire sur l'application du calcul des probabilités aux operations du nivellement topographique.' *Mém. Acad. Belgique,* Vol. XXI, 25 pp.
An excellent practical paper.

1848 WOLF. Bestimmung mittlerer Längen und Gewichte. *Mittheil. Gesell. Bern* for 1848, pp. 238–243.

1849 BAEYER. *Die Küstenvermessung und ihre Verbindung mit der Berliner Grundlinie.* Berlin, 4to.
A valuable work for geodetic engineers.

1849 GÖTZE. 'Ueber die Bestimmung der Bedingungsgleichungen bei der Aufsuchungen der wahrscheinlichsten Elemente eines Himmelskörpers aus einem vollständigen Systeme von geocentrischen Beobachtungen.' *Astron. Nachr.*, Ergänzungs Heft, col. 159–234, 239–240.

1849 PAUCKER. Der Ausgleichungsbau und der mittlere Fehler der Beobachtung. *Arbeit Kürland. Gesell.*, Vol. VII, pp. 91–131.

1849 WITTSTEIN. 'Die Methode der kleinsten Quadrate.' Appendix to *Wittstein's Lehrbuch der Differential-und-Integralrechnung* (Hannover, 1868–9, 8vo), Vol. II, pp. 343–442.
Contains a very clear exposition of HAGEN's demonstration (1837) of the law of facility of error and an excellent elementary presentation of the practical features of the method.

1849 WOLF. Note zur Methode der kleinsten Quadrate. *Mittheil. Gesell. Bern* for 1849, pp. 140–144.

1849 WOLF. Versuche zur Vergleichung der Erfahrungswahrscheinlichkeit mit der mathematischen Wahrscheinlichkeit. *Mittheil. Gesell. Bern* for 1849, pp. 97–100, 183–185; for 1850, pp. 85–88, 209–212; for 1851, pp. 17–36.
See also 1853 WOLF. I regret that I have been unable to see these articles.

1850 BACHE. 'Comparison of the Results obtained in Geodesy by the Application of the Theory of Least Squares.' *Proc. Amer. Assoc.* for 1849, pp. 102–105.

A statement of probable errors of measurements of angles in U. S. Coast Survey triangulations. At the end of the paper, which seems to be a brief abstract of the original, there are some remarks by PEIRCE, GOULD and HENRY, which probably were incorrectly reported.

1850 ENCKE. 'Ueber die Anwendung der Wahrscheinlichkeits-Rechnung auf Beobachtungen.' *Berlin Astron. Jahrb.* for 1853, pp. 310–351.

The object of this paper is to establish greater confidence in the practice of taking the arithmetical mean and in the validity of the exponential law of facility of error. Six of the ten problems of LAGRANGE's memoir of 1774 are translated and a few comments added. HAGEN's demonstration of 1837 is also given in full and spoken of in very favorable terms. ENCKE alludes to the use of the "Erfahrungssatz des Prinzips des arithmetischen Mittels" in his memoir of 1832 and says, "so blieb doch immer eine willkührliche Annahme übrig." At the end of the article is an attempt to explain why $\int_{-\infty}^{+\infty} \varphi(x)dx = 1$, when $\varphi(x)$ is the probability of the error x.

1850 GUY. 'On the Relative Value of Averages derived from different Observations.' *Jour. Statis. Soc.*, Vol. VIII, pp. 30–45.

The observations discussed are statistical facts.

1850 HERSCHEL. 'QUETELET on Probabilities.' *Edinburgh Rev.*, Vol. XCII, pp. 1–57. —*Herschel's Essays* (London, 1857, 8vo), pp. 365–465.

This paper contains in a popular form another proof of the Method of Least Squares. Supposing a stone dropped with the intention that it shall hit a mark on a horizontal plane, the reasoning assumes that the deflections from rectangular axes through the mark are independent; and deduces the exponential form $ce^{-h^2x^2}$ for the law of deviation or error. From this the Method of Least Squares at once follows. This proof was put into algebraic language by ELLIS (see below) and the unwarrantable character of the assumption clearly pointed out. See above 1846 BRAVAIS, and below 1857 BOOLE, 1867 THOMPSON and TAIT, 1872 SCHLÖMILCH, and particularly 1872 GLAISHER. See also 1808 ADRAIN, where this proof was first given.

1850 ELLIS. 'Remarks on an alleged Proof of the "Method of Least Squares," contained in a late number of the *Edinburgh Review*.' *Lond. Phil. Mag.*, Vol. XXXVII, pp. 321–328, 462. —Also *Ellis's Mathematical Writings* (Cambridge, 1863, 8vo), pp. 51–62.

Herschel's proof is discussed and regarded as unsatisfactory. Laplace's method is also explained and defended. The paper is very interesting and valuable. See Glaisher, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, pp. 112–113; also below 1851 Donkin.

1850 Hossard and Poudra. 'Seconde Théorème de minimum pour un système de droite dans l'espace.' *Nouv. Ann. Math.*, Vol. IX, pp. 241–242.

On the centre of gravity of a system of points. See also Vol. VII, p. 407 and 454.

1850 Paucker. Der mittlere Fehler zweiter Ordnung. *Arbeit Kürland. Gesell.*, Vol. VIII, pp. 104–112.

1850 Verdam. '*Verhandeling over de methode der kleinste quadraten.*' Groningen, Vol. I, 4to, pp. xxi, 409.

I have not seen Vol. II of this work and find no direct reference to its publication: it was probably published about 1853. Vol. I is devoted to the practical features of the method and Vol. II to the theory.

Vol. I assumes the principle of Least Squares and develops the methods of adjustments and comparison by probable errors, weights, etc. Numerous well chosen practical examples are given worked out in detail. The work abounds in historical information, and is the most complete text-book on the subject which has come to my notice.

1851 Airy. 'On a Question of Probabilities which occurs in the use of a fixed collimator for the Verification of the Constancy of Position of an Azimuth Circle.' *Rep. Brit. Assoc.* for 1850, pp. 1–2.

Is it justifiable to use the Theory of Errors in finding the probable errors of quantities which are partly results of observation and partly deductions of formulæ?

1851 Airy. 'On the Weights to be given to the separate Results for Terrestrial Longitudes, determined by Observations of Transits of the Moon and Fixed Stars.' *Mem. Astron. Soc. Lond.*, Vol. XIX, pp. 213–229.

Weight is defined as "the reciprocal of the square of the probable error." The paper discusses nine practical cases.

1851 Denzler. 'Ueber den Fundamentalsatz der Methode der kleinsten Quadrate.' *Mittheil. Gesell. Zürich*, Vol. II, pp. 110–118.

Appears to be of little importance.

1851 Donkin. 'On certain Questions relating to the Theory of Probabilities.' *Lond. Phil. Mag.*, Vol. I, pp. 353–368, 458–466; Vol. II, pp. 55–66.

Part III of this paper offers some critical remarks on the Theory of Least Squares with particular reference to ELLIS's paper of 1850. HERSCHEL's proof, it is said, "should be treated with respect." The Method of Least Squares may be used, if for no other reason, because "it is a very *good* method," as shown by GAUSS's proof of 1823.

1851 HOSSARD. 'Note sur la méthode des moindres carrés.' *Nouv. Annal. Math.*, Vol. X, pp. 456–460.

IVORY's first proof (1825) is here rediscovered under a slightly different form.

1851 PAUCKER. Uebereinstimmung der ausgeglichenen Ursachen mit den durch BESSEL's Verfahren gefundenen. *Arbeit Kürland. Gesell.*, Vol. IX, pp. 170–183.

1851 PAUCKER. Einfluss der Gewichte auf die Ausgleichung. *Arbeit Kürland. Gesell.*, Vol. IX, pp. 183–193.

The substance of this and the preceding article is given in the following.

1851 PAUCKER. 'Zur Theorie der kleinsten Quadrate.' *Bull. phys. math. Acad. St. Péters.*, Vol. IX, col. 113–125; Vol. X, col. 33–43, 233–238. —*Mél. math. Acad. St. Péters.*, Vol. I, pp. 188–204, 333–346, 433–439.

Contains new methods of computation, tests of accuracy, etc., which appear to be of little value.

1852 BIENAYMÉ. 'Mémoire sur la probabilité des erreurs d'après la méthode des moindres carrés.' *Liouville's Jour. Math.*, Vol. XVII, pp. 33–78. —*Mém.. par divers savans.. Inst. France*, Vol. XV, pp. 615–663.

After some interesting critical remarks, LAPLACE's analysis (1812) is given considerably simplified. According to BIENAYMÉ's investigation the formulæ for probable error ordinarily used are only correct for one unknown quantity. For two, three and four unknown quantities, he finds that the probable errors should be respectively 1.746, 2.281 and 2.716 times larger than those given by the usual formulæ. His expression for the probability that an error is included between given limits differs sensibly for several unknown quantities from the common probability integral, particularly for limits but little removed from $x=0$. See 1873 WREDE.

See *Comptes Rendus Acad. Paris*, Vol. XXXIV, pp. 90–92, or *Liouville's Jour. Math.*, Vol. XVII, pp. 31–32 for a report on this memoir. See also MEYER's *Calcul des Probabilités*, pp. 377–408.

1852 BIVER. *Théorie des moindres carrés établie par l'analyse pure.* Bruxelles, 8vo.

Probably similar to his memoir of 1853.

1852 DIENGER. 'Ueber die Ausgleichung der Beobachtungsfehler.' *Archiv. Math. u. Phys.*, Vol. XVIII, pp. 149–193; Vol. XIX, pp. 211–227.

HAGEN's demonstration (1837) is followed. The article forms an almost complete elementary treatise on the Method of Least Squares. In the Supplement HAGEN's proof is abandoned, as resting on a questionable hypothesis and GAUSS's first proof is given in its place. DIENGER appreciates clearly the defects of GAUSS's method, for he requires the number of observations to be infinite in order that the value given by the arithmetical mean shall coincide with the *true* value of the measured quantity.

1852 DIENGER. 'Ueber die Bestimmung des Gewichts der nach der Methode der kleinsten Quadrate erhaltenen wahrscheinlichsten Werthe der Unbekannten, wenn Bedingungsgleichungen vorhanden sind.' *Archiv. Math. u. Phys.*, Vol. XIX, pp. 197–202.

Contains valuable practical formulæ for the computer.

1852 HARTNER. *Handbuch der niedern Geodäsie, nebst einem Anhange über die Elemente der Markscheidekunst.* Wien, 8vo. —Second edition, 1856, 8vo, pp. xvi, 611. —Fourth edition, 1872, 8vo.

See 1863 BÖRSCH.

1852 LIAGRE. 'Sur la valeur la plus probable d'un côte géodésique commun à deux triangulations.' *Bull. Acad. Belgique*, Vol. XIX, Pt. I, pp. 513–534.

A clear exposition and solution of the problem. An example from 1849 BAEYER is discussed.

1852 LIAGRE. 'Sur la loi de répartition des hauteurs barométriques, par rapport à la hauteur moyenne.' *Bull. Acad. Belgique*, Vol. XIX, Pt. II, pp. 502–514.

The law is shown to agree with the exponential law of facility of error.

1852 PEIRCE (B.) 'Criterion for the Rejection of doubtful observations.' *Gould's Astron. Jour.*, Vol. II, pp. 161–163.

This Criterion, founded on a principle of the Theory of Probability, proposes a method for determining by successive approximation, whether or not a suspected observation may be rejected. Tables are needed for its application: for these see below 1855 GOULD and 1864 CHAUVENET.

It is a fatal objection to this criterion that its use involves a contradiction of reasoning. The arithmetical mean, for instance, can only be used when the observations are all of equal weight, and the rejection of an observation which deviates considerably from the mean asserts that the weights of the several values are not equal. See below 1856 AIRY and WINLOCK, 1868 STONE and particularly 1872 and 1873 GLAISHER.

The criterion has been used to some extent in the U. S. Coast Survey office, but has elsewhere, I believe, found no acceptance.

1852 QUETELET. 'Sur quelques proprietés curieuses qui preséntent les résultats d'une série d'observations, faites dans la vue de déterminer une constant, lorsque les chances de rencontrer des écarts en plus et en moins soint égales et indépendantes les unes des autres.' *Bull. Acad. Belgique*, Vol. XIX, Pt. II, pp. 303–317.

An interesting investigation, illustrated by a discussion of meteorological observations.

1852 WOLF. Beitrag zur Lehre von der Wahrscheinlichkeit. *Mittheil. Gesell. Bern* for 1852, pp. 133–134.

1853 BIVER. 'Théorie analytique des moindres carrés.' *Liouville's Jour. Math.*, Vol. XVIII, pp. 169–200.

The principle of the arithmetical mean is proved according to 1832 ENCKE. The term "risque de erreur" is given to the function $A+B\Sigma x^2+C\Sigma x^4 D+\Sigma x^6+\ldots$ and it is shown that this becomes a minimum when Σx^2 is a minimum, and this condition is regarded as furnishing "les valeurs les plus plausibles des inconnues." Formulæ for weights and mean errors are also developed.

1853 CAUCHY. 'Mémoire sur l'evaluation d'inconnues déterminées par un grand nombre d'equations approximatives du premier degré.' *Comptes Rendus Acad. Paris*, Vol. XXVI, pp. 1114–1122.

It is maintained that the method of interpolation (1835 CAUCHY) can be used for determining several unknown quantities from a redundant number of equations, with results nearly as accurate as by the Method of Least Squares.

1853 BIENAYMÉ. 'Remarque sur les differences qui distinguent l'interpolation de M. CAUCHY de la méthode des moindres carrés, et qui assurent la supériorite de cette méthode.' *Comptes Rendus Acad. Paris*, Vol. XXVII, pp. 5–13. —*Liouville's Jour. Math.*, Vol. XVIII, pp. 299–308.

It is maintained that the two methods differ "complétement," and that even a contradiction exists. CAUCHY's method, it is said, is only a modification of the ordinary process of elimination, which assures no especial degree of probability to the results and which requires in practice as many operations as the Method of Least Squares. See below CAUCHY.

Further remarks by Bienaymé referring to this discussion are given in pages 68–69, 197, 206 of Vol. XXXVII of the *Comptes Rendus.*

1853 Cauchy. 'Mémoire sur l'interpolation, ou Remarques sur les Remarques de M. Jules Bienaymé.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 64–68.

Gives an extract from the memoir of 1835, and maintains that in many investigations the method of interpolation is preferable to that of Least Squares.

1853 Cauchy. 'Sur la nouvelle méthode d'interpolation comparée à la méthode des moindres carrés.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 100–109.

The new method is claimed to be often the shortest, and the Method of Least Squares is said to give most probable results only under certain conditions.

1853 Cauchy. 'Mémoire sur les coefficients limitateurs ou restricteurs.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 150–162.

In the latter part of the article the "restricteurs" are applied to the theory of Least Squares, and it is concluded that that Method furnishes most probable results only when the law of facility of error is the same for all the errors, when no limits can be assigned to the magnitude of an error, and when the probability of an error x is proportional to $e^{-h^2x^2}$.

1853 Cauchy. 'Sur les résultats moyens d'observations de même nature, et sur les résultats les plus probables.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 198–206.

The conclusions of the preceding article are confirmed.

1853 Cauchy. 'Sur la probabilité des erreurs qui affectent des résultats moyens d'observations de même nature.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 264–272.

Shows that the most probable values may sometimes differ from those found by the Method of Least Squares.

1853 Bienaymé. 'Considerations à l'appui de la découverte de Laplace sur la loi des probabilités dans la méthode des moindres carrés.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 309–324. —*Liouville's Jour. Math.*, 1867, Vol. XII, pp. 158–176.

An answer and review of some of Cauchy's articles: also maintains that the mean of the sum of the squares of the errors is under all circumstances a measure of the precision of the observations.

1853 Cauchy. 'Sur la plus grande erreur à craindre dans un résultat moyen, et sur le système de facteurs qui rend cette plus grande erreur un minimum.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 326–334.

The system of factors is often very different from that given by the Method of Least Squares.

1853 Cauchy. 'Mémoire sur les résultats moyens d'un très grand nombre d'observations.' *Comptes Rendus Acad. Paris*, Vol. XXXVII, pp. 381–385.

An abstract only is given. The result seems to be that the mean is worthy of great confidence.

1853 Grunert. 'Elementare Betrachtungen über die Bildung der Bedingungsgleichungen aus gegebenen Beobachtungen.' *Archiv Math. u. Phys.*, Vol. XXI, pp. 453–486.

"Bedingungsgleichung" is not here used in its usual sense. The paper contains an investigation of the value to be taken for x when $A-Bx=0$, and many values of A and B are given by observation. A certain form for x is shown to involve the principle of Least Squares.

1853 Liagre. *Calcul des probabilités et la théorie des erreurs, avec des applications aux sciences d'observation en general, et à la géodésie en particular.* Bruxelles, 8vo.

This is a standard work on the subject.

1853 Liagre. 'Sur l'erreur probable d'un passage observé à la lunnette méridienne de l'Observatoire royal de Bruxelles.' *Bull. Acad. Belgique*, Vol. XX, Pt. II, pp. 303–312.

1853 Quetelet. *Théorie des probabilités.* Bruxelles, 12mo, pp. 104.

1853 Wolf. Versuche zur Vergleichung der mathemathischen Wahrscheinlichkeit. *Mittheil. Gesett. Bern* for 1853, pp. 23–28.

See also 1849 Wolf.

1853 ———. *Tafeln zur Berechnung der Wahrscheinlichkeit des Vorkommens von Beobachtungsfehler.* Berlin, lith. MS., 4to, pp. 11.

1854 d'Arrest. 'Beitrag zur Methode der kleinsten Quadrate.' *Bericht. Sächsich. Gesell.*, 1854, pp. 133–136. —*Astron. Nachr.*, Vol. XLI, col. 35–40.

Points out six new geometrical properties of the probability curve, and shows how its equation may be derived from a certain mechanical idea.

1854 Hélie. *Mémoire sur la probabilité du tir des projectiles de l'artillerie navale.* Paris, 4to. —Second ed., 1856, 8vo.

1855 Bache. 'Comparison of the reduction of horizontal angles by the methods of dependant directions and of dependant angular quantities, by the method of least squares.' *Rep. Coast Survey U. S.* for 1854, pp. 63*–70.*

The first method is shown to possess a slight advantage.

1855 Bertrand. '*Méthode des moindres carrés. Mémoires sur la combinaison des observations, par* Ch. Fr. Gauss. Paris, 8vo, pp. 167.

This volume renders quite accessible the Latin memoirs of Gauss. It contains translations of pages 208–220 of the *Theoria motus* ..... 1809, pages 20–26 of *Disquisito de* ..... *Palladis*, 1811, and the whole of the memoirs of 1816, 1822, 1823, 1826 and 1827. The memoirs of 1823 and 1827 form the bulk of the book, the others being added at the end as "Notes."

1855 Bertrand. 'Sur la méthode des moindres carrés.' *Comptes Rendus Acad. Paris*, Vol. XL, pp. 1190–1192.

Historical and critical remarks made on presenting a copy of the above book to the Paris Academy.

1855 Gerling. 'Ueber die Schätzung des mittlern Fehlers directer Beobachtungen.' *Archiv Math. u. Phys.*, Vol. XXV, pp. 219–222.

Contains three pages of corrections and errata to his book (1843).

1855 Gould. 'Report ...... containing directions and tables for the use of Peirce's Criterion for the Rejection of Doubtful Observations.' *Rep. Coast Survey U. S.* for 1854, pp. 131*–138*.

Reprinted with a different title in *Gould's Astron. Jour.*, 1855, Vol. IV, pp. 81–87.

1855 Hilgard. 'Discussion of the probable error of observation with a twenty-six inch portable transit ......' *Rep. Coast Sur. U. S.*, for 1854, p. 121*.

The probable error is given as $\pm 0.072^s$.

1855 Liagre. 'Sur la probabilité de l'existence d'une cause d'erreur regulière dans une série d'observations.' *Bull. Acad. Belgique*, Vol. XXII, Pt. II, pp. 9–13, 15–54.

A valuable theoretical discussion illustrated with practical examples.

1855 LLOYD. On the mean results of observations. *Trans. Irish Acad.*, Vol. XXII, pp. 61–73.

See abstract in *Proc. Irish Acad.*, Vol. IV, pp. 180–183.

1855 PEIRCE (B.). 'Report upon the Determination of Longitude by Moon Culminations.' *Rep. Coast Sur. U. S.* for 1854, pp. 109*–120*.

"The small errors which are beyond the limits of human perception, are not distributed according to the mode recognized by the Method of Least Squares, but either with the uniformity which is the ordinary characteristic of matters of chance, or more frequently in some arbitrary form dependant upon individual peculiarities . . . . ."

1855 SCHOTT. 'Adjustment of horizontal angles of a triangulation.' *Rep. Coast Sur. U. S.*, for 1854, pp. 70*–86*.

This and many of SCHOTT's following papers are very valuable, but they are not usually clear except to those who already understand the subject.

1855 SCHOTT. 'Probable error of observation derived from observations of horizontal angles at any single station, and depending on directions.' *Rep. Coast Sur. U. S.* for 1854, pp. 86*-95*.

A discussion of 350 measurements taken at eleven stations.

1856 AIRY. 'Letter from . . . [Remarks on PEIRCE's Criterion.]' *Gould's Astron. Jour.*, Vol. IV, pp. 137–138.

The Criterion is strongly opposed. ". . . . the whole theory is defective in its foundations and illusory in its results." It must be said, however, that some of AIRY's objections are not supported by very good logic.

1856 WINLOCK. 'On Professor AIRY's objections to PEIRCE's Criterion.' *Gould's Astron. Jour.*, Vol. IV, pp. 145–147.

AIRY's objections are taken up in detail; some of them are shown to apply equally well to the Method of Least Squares.

1856 PETERS. 'Ueber die Bestimmung des wahrscheinlichen Fehlers einer Beobachtung aus den Abweichungen der Beobachtungen von ihrem arithmetischen Mittel.' *Astron. Nachr.*, Vol. XLIV, col. 29–32. —Trans. in *Rep. Coast Sur. U. S.* for 1856, pp. 307–308.

Let Σx be the sum of the residual errors all taken positive, and n the number of direct observations of equal weight. Then PETERS' result is, that r, the probable error of a single observation is,

$$r = 0.845347 \frac{\Sigma x}{\sqrt{n(n-1)}}.$$

See on this formula 1869 LÜROTH, and 1875 HELMERT.

1856 SCHOTT. 'Solution of Normal Equations by indirect Elimination.' *Rep. Coast Sur. U. S.* for 1855, pp. 255–264.

Several methods are given and illustrated by numerical examples. The paper is of great value to a computer.

1856 SCHOTT. 'Report . . . . . of a discussion of the secular variation in the magnetic declination on the Atlantic and part of the Gulf Coast of the United States.' *Rep. Coast Survey U. S.* for 1855, pp. 306–337.

The process of deducing empirical formulæ for declination from observations by the Method of Least Squares is explained and illustrated, as also that of finding the probable errors of the constants which enter into such formulæ and of the computed results. Formulæ are deduced for fourteen stations. See also *Rep. Coast Survey* for 1855, p. 306, and for 1859, p. 296.

1856 VORLÄNDER. 'Ueber die Genauigkeit der Längenmessungen mit der Messkette auf verschiedenen Bodenarten.' *Zeitschr. Math. u. Phys.*, Vol. I, pp. 142–159.

The precision is regarded as inversely proportional to the length of the line. Tables are given showing results for different kinds of ground. See 1863 BÖRSCH.

1857 ANDRÄ. 'Fehlerbestimmung bei der Auflösung der POTHENOT'schen Aufgabe mit dem Messtische.' *Astron. Nachr.*, Vol. XLVII, col. 193–202.

Points for which the probabilities of error are equal have an ellipse as locus. The most probable ellipse is assumed to be given by the Method of Least Squares. See *Bull. math. phys. Acad. St. Peters.*, Vol. VII, p. 145.

1857 D'ARREST. 'Schreiben . . . . [über die Berechnung der Gewichte].' *Astron. Nachr.*, Vol. XLVII, col. 17–20.

Contains formulæ for the easy determination of the relative weights of three unknown quantities, and also remarks concerning the geometric signification of weights and their connection with determinants.

1857 BABINET and HOUSEL. '*Calculs pratiques appliqués aux sciences d'observation.*' Paris, 8vo, pp. xvi, 388.

Mostly devoted to the theory of numerical approximations and interpolation formulæ. Two pages are given to the Method of Least Squares.

1857 BAUR. —————————. *Programm des Stuttgarter Polytechnikursus* für 1857.

An article on determination of weights, etc.

1857 BELLAVITIS. Considerazioni sulla theoria della probabilita. *Atti Istit. Venezia* for 1856–7, pp. 299–321.

1857 BOND. 'On the use of Equivalent Factors in the Method of Least Squares.' *Mem. Amer. Acad.*, Vol. VI, pp. 179–212. —Also privately printed, Cambridge, 1858, 4to, pp. 36.

It is shown that it is unnecessary to adhere in numerical computations to the strict letter of the Method of Least Squares, and that its application "........requires the use of such numbers only, in the arithmetical processes peculiar to it and characteristic of the Method, as may be designated by one of the numerals 0, 1, 2,....9, or of the fractions $\frac{1}{2}$, $\frac{1}{3}$,....$\frac{1}{9}$, or by a product of one of these numbers by an integral power of 10."

1857 BOOLE. On the Application of the Theory of Probabilities to the question of the Combination of Testimonies or Judgments. *Trans. Soc. Edinb.*, Vol. XXI, pp. 597–652.

In the first part of this memoir the rule of the arithmetical mean is discussed. "The result of BOOLE's investigation is that if n observations $p_1, p_2, \ldots p_n$ be made upon the same quantity, then the most probable value of that quantity is a certain linear function of p_1, p_2, ...p_n; this BOOLE demonstrates by his Calculus of Logic, and the analysis is of so peculiar a character that....I feel scarcely qualified to express a decided opinion on its merits. ....the [final] result takes the form of the arithmetic mean."—GLAISHER, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, p. 124.

In the latter part of the paper HERSCHEL's demonstration is reproduced and defended against the arguments of ELLIS; see 1850. See GLAISHER's paper, just quoted, pp. 115.

1857 DIENGER. '*Ausgleichung der Beobachtungsfehler nach der Methode der kleinsten Quadratssummen. Mit zahlreichen Anwendung, namentlich auf geodätische Messungen.*' Braunschweig, 8vo, pp. viii, 168.

An excellent elementary text-book. GAUSS's proof of 1809 is followed, with the improvement that the probability of a definite error is an infinitesimal. See 1852 DIENGER. Among the practical questions treated is the theory of repetitions in angle measurements.

1857 DONKIN. 'On an Analogy relating to the Theory of Probabilities, and on the Principle of the Method of Least Squares.' *Quart. Jour. Math.*, Vol. I, pp. 152–162.

Donkin observes that if two observations of an unknown quantity give $x=a$ and $x=b$, then the most probable value of x is $\frac{1}{2}(a+b)$, but that we cannot regard the arithmetical mean of more than two observations as most probable. Taking x to represent the true value of the unknown quantity DONKIN says: "....it appears a natural and obvious assumption (though I do not pretend that it is *not* an assumption) that *the probability that x is between x and $x+dx$ must be expressible in the form* $\psi\left(x-\frac{a+b}{2}\right)dx$." From this the exponential law of facility of error is deduced.

For an analysis of this reasoning see GLAISHER, *Mem. Astron. Soc. Lond.*, Vol. XXXIX, p. 118, who classes it among the proofs of the Method.

1857 LIAGRE. Sur la mesure de précision des observations méridiennes faites à l'Observatoire royal de Bruxelles. *Bull. Acad. Belgique*, Vol. III, pp. 330–338.

1857 PETZVAL. 'Fortsetzung des Berichtes über optische Untersuchungen. [Ausgleichungstheorie].' *Sitzungsber. math-nat. Acad. Wien*, Vol. XXIV, pp. 129–144.

PETZVAL concludes that the Method of Least Squares is entirely inapplicable in Optics. He proposes "die Methode der numerisch gleichen Maxima und Mimmia," which consists in making the sum of the $2m$ powers a minimum, m being a variable which tends toward infinity as a limit. The development and application of this method is to constitute the First Part of Vol. III of his work on Optics. This method was mentioned by LAPLACE in the *Théorie....des Probabilités*, p. 345.

1857 SAWITSCH. *Die Anwendung der Wahrscheinlichkeitstheorie auf die Berechnung der Beobachtungen und geodätische Messungen, oder die Methode der kleinsten Quadrate.* (Russian). St. Petersburg, 8vo. —German edition, Leipzig, 1863, 8vo.

1857 VORLÄNDER. 'Ueber das geodätische Vorwärts-Einschneiden.' *Zeitschr. Math. u. Phys.*, Vol. II, pp. 299–316.

A practical application of the Method of Least Squares.

1857 WREDE. 'Några anmärkuingar rörande minsta quadratmethodens tillämpning.' *Öfvers. Acad. Förhandl. Stockholm*, Vol. XIV, pp. 73–81.

On the error ellipse.

1857 ZECH. *Einladung zur akademischen Feier des Geburtstags des Königs von Württemberg, nebst einer Abhandlung zur Methode der kleinsten Quadrate.* Tübingen, 4to.

The first exhaustive discussion of the adjustment of indirect observations subject to conditional equations. See JORDAN, *Elemente der Vermessungskunde* (Stuttgart, 1877, 8vo), p. 6.

1858 BOUNIAKOWSKY. 'Sur un instrument destiné à faciliter l'application numérique de la méthode des moindres carrés, et à contrôler les résultats obtenus par cette méthode.' *Bull. phys. math. Acad. St. Péters.*, Vol. XVII, col. 289–298. —*Mél. math. Acad. St. Péters.*, Vol. II, pp. 602–614.

A machine for calculating the sums $a_1^2 + a_2^2, + a_3^2 + \dots$, and $a_1b_1 + a_2b_2 + a_3b_3 + \cdots$

1858 CASORATI. 'Intorno ad alcuni puncti della theoria dei minimi quadrati.' *Annali di Math.*, Vol. I, pp. 329–343.

Discussion by the use of determinants, etc.

1858 CLARKE. '*Ordnance Trigonometrical Survey of Great Britain and Ireland. Account of the Observations and Calculations of the Principal Triangulations and of the Figure, Dimensions and mean Specific Gravity of the Earth as derived therefrom.*' London, 4to, pp. xvii, 782, with an Atlas of 28 Plates.

The whole triangulation is adjusted by the Method of Least Squares. The method of correlatives is explained at length and illustrated for a case involving seventy-four observations subject to thirty-nine conditional equations. An inspection of this book will give students an idea of the stupendous calculations which men of science undertake and execute.

1858 DIDION. *Calcul des probabilités appliqué au tir des projectiles.* Paris, 8vo.

For an exposition of this subject see SONNET's *Dictionaire des mathématiques appliquées*, (Paris, 1867), pp. 1103–1108.

1858 GRUNERT. 'Drei Grössen x, y, z, deren Summe die gegebene Grösse s ist, sind durch Messung bestimmt worden, und man habe dadurch für diese drei Grössen respective die Werthe a, b, c erhalten. Da diese Werthe mit Beobachtungsfehler behaftet sind, und ihre Summe also im Allgemeinen nicht genau s ist, so soll man dieselben so verbessern, dass die verbesserten Werthe genau die Summe s geben, und die Summe der Quadrate der Verbesserungen ein Minimum ist.' *Archiv. Math. u. Phys.*, Vol. XXXI, pp. 480–481.

1858 JULLIEN. 'Mémoire sur la probabilité des erreurs dans la somme ou dans la moyenne de plusieurs observations.' *Annali di Math.*, Vol. I, pp. 76–88, 149–155, 227–237.

Errors subject to the law of facility $\varphi(x) =$ constant are particularly discussed.

1858 KOHLER. '*Die Landesvermessung des Königsreichs Württemberg.*' Stuttgart, 8vo, pp. xii, 428, and 3 pl.

In the Appendix particularly are applications of the Method of Least Squares.

1858 RITTER. '*Manual théoretique de l'application de la méthode des moindres carrés au calcul des observations.*' Paris, 8vo, pp. 80.

An excellent little text-book, in which GAUSS's first demonstration is followed, with the improvement that the probability of a single error is an infinitesimal.

1858 SCHINDLER. *Ueber Fehler bei der Berechnung eines ebenen Dreiecks.* Prag, 4to.

1858 SCHOTT. 'Account of the method and formulæ for the determination of the astronomical latitude by means of the zenith telescope as used in the survey of the coast of the United States.' *Rep. Coast Survey U. S.* for 1857, pp. 324–334.

Weights and probable errors are found. The notation used on p. 333 is unusual and uncouth.

1858 TCHÉBYCHEFF. 'Sur les fractions continues.' *Liouville's Jour. Math.*, Vol. III, pp. 289–325.

Translated from the Russian by BIENAYMÉ: treats of interpolation by the Method of Least Squares.

1858 TCHÉBYCHEFF. 'Sur l'interpolation des valeurs fournis par les observations.' *Bull. phys. math. Acad. St. Péters.*, Vol. XVI, col. 353–357. —*Mél. math. Acad. St. Péters.*, Vol. II, pp. 345–351.

1858 VORLÄNDER. 'Zur praktischen Geometrie.' *Zeitschr. Math. u. Phys.*, Vol. III, pp. 189–193.

On the adjustment of a quadrilateral whose sides and diagonals are measured.

1858 VORLÄNDER. 'Bemerkungen über das numerische Eliminiren bei geodatischen Operationen.' *Zeitschr. Math. u. Phys.*, Vol. III, pp. 16–22.

On methods of abridging the computations.

1858 GERLING. 'Bemerkungen über das indirecte Eliminiren bei geodätischen Arbeiten.' *Zeitschr. Math. u. Phys.*, Vol. III, pp. 377–382.

Referring to the preceding article of VORLÄNDER.

1858 VORLÄNDER. *Ausgleichung der Fehler polygonometrischer Messungen.* Leipzig, 8vo, pp. 55.

Besides the Method of Least Squares shorter approximate processes are given.

1858 DIENGER. 'Ueber die Ermittelung des wahrscheinlichen Fehlers bei Längenmessungen.' *Archiv. Math. u. Phys.*, Vol. XXXI, pp. 225–228.

Opposes VORLÄNDER's method given in the preceding.

1859 TCHÉBYCHEFF. 'Sur l'interpolation dans le cas d'un grand nombre de données fournies par les observations.' *Mém. Acad. St. Péters.*, Vol. I, No. 5, 81 pp.

1859 TCHÉBYCHEFF. 'Sur l'interpolation par la méthode des moindres carrés.' *Mém. Acad. St. Péters.*, Vol. I, No. 15, 24 pp.

[1860] BAEYER. *Wissenschaftliche Begründung der Rechnungsmethoden des Centralbureaus der europäischen Gradmessung: I. Die Methode der kleinsten Quadrate. II. Die Anwendung derselben auf Geodäsie.* [Lith. MSS. ?] 4to.

1860 DEDAKIND. 'Ueber die Bestimmung der Präcision einer Beobachtungsmethode nach der Methode der kleinsten Quadrate.' *Vierteljahrs. Gesell. Zürich*, Vol. V, pp. 76–83.

The most probable value of the measure of precision h is found to be $\sqrt{\frac{n-1}{2\Sigma x^2}}$ and not $\sqrt{\frac{n}{2\Sigma x^2}}$. See 1866 BÖRSCH.

1860 HULTMANN. *Sur les moindres carrés.* Stockholm, 4to.

1861 AIRY. '*On the Algebraical and Numerical Theory of Errors of Observations and the Combination of Observations.*' Cambridge and London, 8vo, pp. xvi, 103. —Second ed., 1875.

Only LAPLACE's *Théorie analytique des Probabilités* was consulted in preparing this book, and as a consequence it is unreadable except by those already thoroughly acquainted with the subject.

1861 BORCHARDT. 'Ueber Interpolation nach der Methode der kleinsten Quadrate.' *Crelle's Jour. Math.*, Vol. LVIII, pp. 270–272.

On TCHÉBYCHEF's method; see 1858 and 1859.

1861 GERLING. Notiz in Betreff der Prioritäts-Verhältnisse in Beziehung auf die Methode der kleinsten Quadrate. *Nachr. Gesell. Göttingen* for 1861, pp. 273–275.

1861 PETERS. *Ueber die Bestimmung des Längenunterschieds zwischen Altona und Schwerin.* Altona, 4to.

Contains among other matter a detailed history of the discovery of the personal equation. See 1866 RADAU.

1861 SCHOTT. 'Account of CAUCHY's interpolation formula.' *Rep. Coast Sur. U. S.* for 1860, pp. 392–396.

This is a free translation of CAUCHY's article of 1835. The method is illustrated by an example.

1861 WINCKLER. 'Ueber den mittlern Fehler der Kettenmessungen.' *Zeitschr. Math. u. Phys.*, Vol. VI, pp. 109–119.

An excellent practical paper. The mean error is found to be proportional to the square root of the length of the line.

1862 GERLING. 'Ueber Genauigkeit der Functionen bedingter Beobachtungen.' *Archiv. Math. u. Phys.*, Vol. XXXVIII, pp. 379–381.

Contains additions and corrections to his book (1843).

1862 SEIDEL. Ueber eine Anwendung der Wahrscheinlichkeitsrechnung bezüglich auf der Schwankungen in den Durchsichtigkeitsverhältnissen der Luft. *Sitzungsber. Baiersch. Acad.*, Vol. II, pp. 320–349.

1863 BÖRSCH. 'Ueber die Genauigkeit der Winkel- und Linien-Messungen.' *Zeitschr. Math. u. Phys.*, Vol. VIII, pp. 321–341.

It is concluded that the precision of angle measurements is proportional to the square root of the number of single observations, or to the number of repetitions; and that the precision of linear measurements is inversely proportional to the square root of the length of the line. The articles of VORLÄNDER (see 1856) are discussed as also is HARTNER's (1852) treatment of this subject.

1863 FREEDEN. '*Die Praxis der Methode der kleinsten Quadrate für die Bedürfnisse der Anfänger bearbeitet.*—Erster Theil: Elementare Darstellung der Methode nebst Sammlung vollständig berechneter physikalischer, meteorologischer, geodätischer und astronomischer Aufgaben, welche auf lineäre und transcendente Gleichungen führen.' Braunschweig, 8vo, pp. viii, 114.

An excellent little book, although some of the examples are rather long for a beginner. The principle of Least Squares is assumed.

1863 KURZ. 'Ueber der Methode der kleinsten Quadrate.' *Vierteljahrsschr. Gesell. Zürich*, Vol. VIII, pp. 225–240.

On the various methods of forming and solving normal equations, of determining weights, etc.

1863 LOBATTO. Over de waarschijnlijkheid van gemiddelde uitkomsten uit een groot aantal waarnemingen. *Archief. Wisk. Genoots*, Vol. II, pp. 96–127.

1864 CHAUVENET. 'Method of Least Squares.' Appendix to *Manual of Spherical and Practical Astronomy* (Philadelphia, 8vo), Vol. II, pp. 469–566. —Also as separate issue under title '*A Treatise on the Method of Least Squares,....*' Philadelphia, 1868.

This is mainly an abridgment of ENCKE's memoirs of 1832. ENCKE's demonstration of the rule of the arithmetical mean is in particular set forth with confidence. The reasoning showing that if $\varphi(x)$ is the probability of the error x, $\varphi(x)dx$ is the rigorous probability that an error falls between x and $x+dx$ is very illogical. At the end are valuable tables, two of the probability integral, and others for using PEIRCE's criterion which is given nearly in the words of its author; see 1852 and 1856. CHAUVENET adds an approximate criterion for the rejection of *one* doubtful observation, which is derived "directly from the fundamental formula upon which the whole theory of the Method of Least Squares is based."

1864 CHRISTOFFEL. '*Bestimmung einer Oberfläche durch lokale Messungen.* [Berlin], 4to.

1864 DEMORGAN. 'On the Theory of the Errors of Observation.' *Trans. Camb. Phil. Soc.*, Vol. X, pp. 409–427.

This is a very valuable contribution to the theory of the arithmetical mean. It is shown that the average "is not merely the *mean value of all the given values:* it is also the *mean supposition of all possible suppositions* as to the mode of obtaining that value," but that "the average is the most probable result only so long as we *know* nothing of the law of facility of error." See GLAISHER, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, p. 90.

DEMORGAN suggests the name "*critical error*" instead of *probable error.* The entire paper, like all of DEMORGAN's writings, is very interesting and suggestive.

1864 SCHOTT. 'Development of BESSEL's function for the effect of periodic forces, for durations of periods frequently occurring in meteorological and magnetical investigations; with examples.' *Rep. Coast Survey U. S.* for 1862, pp. 232–235.

See above 1828 BESSEL.

1864 Woolhouse. 'On Interpolation, Summation, and the Adjustment of Numerical Tables.' *Lond. Assurance Mag.*, Vol. XI, pp. 61–88, 300–332; Vol. XII, pp. 136–175.

1865 Brünnow. 'The Method of Least Squares.' *Spherical Astronomy* (First English from the second German edition, London and New York, 8vo), pp. 40–60.

A very elementary sketch of the Method.

1865 Gooss. '*Begründung der Methode der kleinsten Quadrate.*' Kreutznach, 8vo, pp. 32.

A doctor's thesis. Contains a deduction of the law $y = ce^{-h^2x^2}$ from the axioms that the curve is symmetrical, that it has the axis of x for an asymtote, that the equation must be a simple one, etc. The discussion is not very satisfactory.

1865 Tait. 'On the Law of Frequency of Error.' *Trans. Soc. Edinb.*, Vol. XXIV, 7 pp.

The principle of the investigation is that an error arising from any source may be compared to the deviation from the most probable result of the number of white or black balls obtained by a great number of drawings from a bag containing equal numbers of white and black balls. The idea and the algebraic work is nearly the same as Quetelet's investigation of 1846. See 1872 Glaisher.

1865 Todhunter. '*A History of the Mathematical Theory of Probability from the time of* Pascal *to that of* Laplace.' Cambridge and London, 8vo, pp. xvi, 624.

This work is invaluable to all students of the Theory of Probability and I have to acknowledge my great indebtedness to it in preparing the early part of this list. None but those who have undertaken such historical researches can form an idea of the immense amount of labor which must have been done in preparing a work like this of Todhunter.

Todhunter's analyses of the memoirs of Lagrange and Laplace are full and clear, and his commentary on Laplace's proof of the Method of Least Squares greatly simplifies the tedious investigations of the *Théorie analytique des Probabilités.* An account of Gauss's proof of 1809 is not given.

1866 Börsch. 'Ueber die mittlern Fehler der Resultate aus trigonometrischen Messungen.' *Archiv. Math. u. Phys.*, Vol. XLVI, pp. 40–44.

Σx^2 being the sum of the squares of the residual errors and n the number of direct observations, the mean error has been taken as

$\sqrt{\frac{\Sigma x^2}{n}}$ and $\sqrt{\frac{\Sigma x^2}{n-1}}$. The paper compares the results given by these two formulæ and accords the preference to the first. See 1816 BESSEL, 1816 GAUSS, 1823 GAUSS and 1860 DEDAKIND.

1866 NATANI. 'Quadrate (Methode der kleinsten).' *Hoffmann's Mathematisches Wörterbuch*, Vol. V, pp. 16–33.

GAUSS's proof of 1808 and HAGEN's of 1837 are given, and free use is made of WITTSTEIN's work of 1849. The familiar equation expressing the law of facility of error appears here under the strange notation $\varphi(\gamma)=Ce^{-a^2\gamma^2}$.

1866 RADAU. 'Ueber die persönlichen Gleichungen bei Beobachtungen derselben Erscheinungen durch verschiedene Beobachter.' *Reper. f. phys. Technik*, Vol. I, pp. 202–218, 306–321; Vol. II, pp. 115–156.

This is a translation from the *Moniteur scientifique* for 1865. It gives a detailed history and discussion of the subject of personal equation. See 1861 PETERS.

1866 SCHOTT. 'The problem of determining a position by angles observed upon a number of given stations. Solution of GAUSS, with example.' *Rep. Coast Survey U. S.* for 1864, pp. 116–119.

The method is taken from GERLING's book, see 1840.

1866 SCHOTT. 'Report on the method of reduction, and results of the connection of the Epping base line with the primary triangulation in the Eastern States.' *Rep. Coast Survey U. S.* for 1864, pp. 120–144.

A valuable practical paper containing determinations of probable errors of observations, and the discussion of a case of adjustment in solving thirty-five normal equations and fifty-eight equations of correlatives.

1866 WINCKLER. 'Allgemeine Sätze zur Theorie der unregelmässigen Beobachtungsfehler.' *Sitzungsber. Akad. Wein*, Vol. LIII, Pt. II, pp. 6–41.

An investigation of relations between mean values of powers of errors and expressions for probability of errors.

1867 ANDRÄ. *Den Danske Gradmaaling.* Vol. I, Kopenhagen, 4to, pp. 608. Vol. II in 1873, pp. 504.

The adjustment of indirect observations subject to conditional equations is fully treated.

1867 HAGEN. '*Grundzüge der Wahrscheinlichkeits-Rechnung.*' Second edition, Berlin, large 8vo, pp. x, 187.

This is rewritten from the edition of 1837. The proof of the law of facility is given substantially the same as before. The distinction between true and computed errors is not however clearly drawn.

The book is an excellent one for students and engineers, the greater part being of a practical character.

1867 HANSEN. 'Von der Methode der kleinsten Quadrate im Allgemeinen und in ihrer Anwendung auf die Geodäsie.' *Abhandl. Sächsisch. Gesell.*, Vol. XIII, pp. 571–806. —Also separately, Leipzig, 8vo, pp. 236.

In the opening pages the law of facility $\varphi(x)=h\pi^{-\frac{1}{2}}e^{-h^2x^2}$ is deduced by GAUSS's method of 1809, x being regarded as the residual or computed error. The remarks on page 797 concerning probable errors seem to be true and valuable. The book is particularly full in the treatment of conditioned observations, and is a valuable one for geodetic engineers. See 1868 and 1869 for supplements to the work.

1867 SCHOTT. 'Results of the primary triangulation of the coast of New England, from the north-eastern boundary to the vicinity of New York.' *Report Coast Survey U. S.* for 1865, pp. 187–203.

A continuation of SCHOTT's articles of 1855 and 1866, devoted mainly to the discussion of the probable errors of the linear and angular measurements of the triangulation. A comparison is also given of the measured lengths of three base lines with the lengths as computed through the triangulation. The paper is a very valuable one.

1867 THOMPSON and TAIT. 'Experience.' Chap. III, of their *Treatise on Natural Philosophy*, (Oxford, 8vo), Vol. I, pp. 303–320.

HERSCHEL's proof (1850) is given and spoken of as "simple and apparently satisfactory." See below 1872 SCHLÖMILCH.

1867 TCHÉBYCHEFF. 'Des valeurs moyens.' *Rec. Sci. Math.*, Vol. II. —*Liouville's Jour. Math.*, Vol. XII, pp. 177–184.

On "Espérances mathématiques," their arithmetical means, etc.

1868 FRISIANI. 'Sulle più vantaggiosa combinazione delle osservazioni.' *Mem. Istit. Lombardo*, Vol. II, pp. 1–21.

The principle of Least Squares is proved, assuming that the arithmetical mean gives the most probable result.

1868 HANSEN. 'Fortgesetzte geodätische Untersuchungen, bestehend in zehn Supplementen zur Abhandlung von der Methode der kleinsten Quadrate im Allgemeinen und in ihrer Anwendung auf die Geodäsie.' *Abhandl. Sächsisch. Gesell.*, Vol. XIV, pp. 1–184.

For review see *Jahrb. Fortschr. Math*, Vol. I, p. 388.

1868 HELMERT. 'Studien über rationelle Vermessungen im Gebiete der höheren Geodäsie.' *Zeitschr. Math. u. Phys.*, Vol. XIII, pp. 73–120, 163–186.

See *Jahrb. Fortschr. Math.*, Vol. I, p. 389.

1868 HENCKE. *Ueber die Methode der kleinsten Quadrate.* Leipzig, 8vo.

A doctor's dissertation containing historical and critical information relating to Least Squares. I regret that I have been unable to see a copy of it.

1868 LEBOULENGÉ. 'Études de ballistique expérimentale.' *Mém. Acad. Belgique*, Vol. XX, pp. 1–94.

Contains applications of the Method of Least Squares.

1868 MILLER-HAUENFELS. '*Höhere Markscheidekunst. Praktisch-theoretische Anleitung beim Markscheiden die vermeidlichen Fehler zu umgehen, die unvermeidlichen aber in einfacher und streng wissenschaftlicher Weise zu verbessern.*' Wien, 8vo, pp. xii, 291.

A valuable book for mining engineers. In the first or practical part processes and their applications are given, while the proofs follow in the second part. An attempt is made to show that the arithmetical mean is the most probable result by the theory of combinations, all true errors being taken as equal. The term *absolute weight* is introduced for h^2. GAUSS's first proof of the Method of Least Squares is given.

1868 SCHIAPARELLI. 'Sul principio della media arithmetica nel calcolo dei resultati delle osservationi.' *Instit. Lombard. Rendiconti*, Vol. I, pp. 771–778.

Contains a new demonstration of the validity of the arithmetical mean. See 1875.

1868 STONE. 'On the Rejection of Discordant Observations.' *Month. Not. Astron. Soc. Lond.*, Vol. XXVIII, pp. 165–168.

PEIRCE's and CHAUVENET's criteria (see 1852 and 1864) are regarded as troublesome to use and as based on an erroneous principle, and a criterion is proposed, which embodies, in the opinion of the author, the true grounds on which the judgment rests when rejecting discordant observations or mistakes. See below 1873 GLAISHER and STONE.

1869 DORNA. 'Sulla media arithmetica vel calcolo di compensazione.' *Atti. Accad. Torino*, Vol. IV, pp. 757–763.

On ENCKE's demonstration of the arithmetical mean, and on SCHIAPARELLI's article of 1868.

1869 FAÀ DE BRUNO. '*Traité élémentaire du calcul des erreurs, avec des tables stéréotypées, ouvrage utile à ceux qui cultivent les sciences d'observation.*' Paris, 8vo, pp. vii, 72, XLV.

The tables are the best part of this *ouvrage*, but in that giving the values of *sin*² there is at least one dangerous error. This was one of the first books on the Method of Least Squares which I read, and I take this opportunity to warn young students against it. The text is full of typographical and other errors and the subject is presented neither clearly or fully. The list of literature at the end does not contain the names of LEGENDRE, IVORY, ENCKE, BESSEL, ELLIS, or HERSCHEL, gives only one work by HANSEN, and does not mention GAUSS's *Theoria motus*..... The book deserves a speedy oblivion.

1869 GALTON. '*Hereditary Genius; an Inquiry into its Laws and Consequences.*' London, 8vo. —Amer. edition, 1870.

The exponential law of error is used in dividing mankind into grades of intellect. QUETELET's numbers (1846) are employed for this purpose and are given in the appendix.

1869 HANSEN. 'Entwickelung eines neuen veränderten Verfahrens zur Ausgleichung eines Dreiecksnetzes mit besonderer Betrachtung des Falles in welchem gewisse Winkel voraus bestimmte Werthe bekommen sollen.' *Abhandl. Sächs. Gesell.*, Vol. XIV, pp. 185–287.

1869 HELMERT. 'Beiträge zur Theorie der Ausgleichung trigonometrischer Netze.' *Zeitschr. Math. u. Phys.*, Vol. XIV, pp. 174–208.

See *Jahrb. Fortschr. Math.*, Vol. II, p. 835.

1869 JORDAN. 'Ueber die Bestimmung der Genauigkeit mehrfach wiederholter Beobachtungen *einer* Unbekannten.' *Astron. Nachr.*, Vol. LXXIV, col. 209–226.

Contains a method for finding probable errors from the $\frac{1}{2}n(n-1)$ differences between n observations taken two by two. See below ANDRÄ.

1869 ANDRÄ. 'Schreiben....[om Bestemmelsen af den sandsynnlige Feil ved Hjaelp af Iagttagelsernes Diffentser].' *Astron. Nachr.*, Vol. LXXIV, col. 283–284.

Objects to JORDAN's method on the ground that the differences are not independent. See 1872 for continuation of this discussion.

1869 LÜROTH. 'Bemerkung über die Bestimmung des wahrscheinlichen Fehlers.' *Astron. Nachr.*, Vol. LXXIII, col. 187–190.

If n be the number of observations and q that of the unknown quantities, the probable error of a single observation is found to be

$$r = 0.8453 \frac{\Sigma x}{\sqrt{n(n-q)}}.$$

This is an extension of the formula given by PETERS in 1856. See 1876 HELMERT.

1869 ROGERS. 'On the Variability of Personal Equation in Transit Observations.' *Amer. Jour. Sci.*, Vol. XLVII, pp. 297–307.

A discussion of interesting experiments.

1869 TODHUNTER. 'On the Method of Least Squares.' *Trans. Camb. Phil. Soc.*, Vol. XI, pp. 219–238.

On page 9 of the First Supplement (1815) or on page 539 of the national edition of the *Théorie....des Prob.*, LAPLACE gave, without demonstration, a certain formula. "The primary object of this communication is to demonstrate the result which as I have stated LAPLACE merely enunciated..... A secondary object of the communication is to develop LAPLACE's own process of investigating the method of Least Squares; some of the results which he obtained for the case of *two* elements are here demonstrated to hold for the case of any number of elements."

1869 WATSON. 'Method of Least Squares, Theory of the Combination of Observations, and Determination of the most probable system of elements from a series of observations.' Chap. VII of his *Theoretical Astronomy* (Philadelphia, 8vo), pp. 360–425.

An elementary sketch of the subject according to GAUSS and ENCKE.

[1869] THIELE. Undersogelse af Omlobsbevaegelsen i Dobelstjerne.

"THIELE hat gezeigt dass der wahrscheinlichste Werthe bei durch Schätzung ermittelnden Doppelsterndistanzen das geometrische Mittel ist."—HELMERT, *Ausgleichungsrechnung*, p. 95.

1870 CROFTON. 'On the Proof of the Law of Errors of Observations.' *Phil. Trans. London* for 1870, pp. 175–188.

The object of this paper is to determine the law of facility of error on the hypothesis that an error arises from the joint operation of a large number of small sources of error, positive and negative errors *not* being equally probable. The investigation is not very clear.

1870 LAZARUS. 'On some problems in the Theory of Probabilities.' *Lond. Assurance Mag.*, Vol. XV, pp. 244–257.

This is translated from the German. It treats of the general term of the binomial $(\frac{1}{2}+\frac{1}{2})^m$ when m is very large.

1870 NEOVIUS. *Lärobok i minsta qvadrat-methoden.* Åbo, 8vo, pp. 109.

For review see *Bull. Math. et Astron.*, Vol. II, pp. 134–136.

1870 SANTINI. Compendiato esposizione del modo più vantaggiosa di resolvere una serie di equazioni lineari, risultanti da operationi tutti ugualmente probabili, per la determinazione degli elementi di una proposta teorica. *Mem. Istit. Veneto.*, Vol. XIV.

1870 TCHÉBYCHEFF. 'Formule d'interpolation par la méthode des moindres carrés.' *Mém. Couronnés Acad. Belgique*, Vol. XXI, (Appendix to "N. Majewski, Mémoire...."), 9 pp.

See *Jahrb. Fortschr. Math.*, Vol. II, p. 116.

1870 WOLF. '*Handbuch der Mathematik, Physik, Geodäsie und Astronomie.*' Zürich, 2 vols. 8vo. Vol. I (1870), pp. xii, 492; Vol. II (1872), pp. viii, 459.

This valuable work of reference contains a brief sketch of the history of the Method of Least Squares, with a short development of its theory according to 1832 ENCKE.

1871 ABBE. 'A Historical Note on the Method of Least Squares.' *Amer. Jour. Sci.*, Vol. I, pp. 411–415.

Points out that the Method was independently discovered and published by ADRAIN in 1808, and reprints a portion of the original investigation. Interesting biographical notes relating to ADRAIN are also given.

1871 FRANKE. '*Die Dreiecksnetze vierter Ordnung, als Grundlagen geodätischer Detail-Aufnahmen zu technischer oder staatswirthschaftlichen Zwecken.*' München, 8vo, pp. xii, 261.

Numerous examples of adjustment are given. The theory and practice of the subject are presented in different chapters. It is an excellent book.

1871 GLAISHER. 'Tables of the Error-function.' *Lond. Phil. Mag.*, Vol. XLII, pp. 431–436.

An account of KRAMP'S, BESSEL'S, ENCKE'S and other tables of the values of the probability integral, with a new table of values from $x=3.00$ to $x=4.50$.

1871 Jordan. Ueber die Genauigkeit einfacher geodätischer Punktbestimmungen. *Zeitschr. für Math. u. Phys.*, Vol. XVI, pp. 397–413.

1871 Klinkerfues. 'Die Berechnung einer Bahn aus einer grösseren Zahl von Beobachtungen nach der Methode der kleinsten Quadrate.' *Theoretische Astronomie* (Braunschweig, 8vo), pp. 328–379.

An elementary sketch of the Method of Least Squares.

1871 Minding. 'Zur Theorie der kleinsten Quadrate.' *Bull. Acad. St. Péters.*, Vol. XVI, col. 305–308.

On the solution of normal equations, determination of weights, etc.

1871 Peters. 'Quadrate der Zahlen von 1 bis 10,000.' *Astron. Tafeln und Formeln* (Leipzig, 8vo), pp. 151–169.

Deserves a place here as an aid in the Method of Least Squares on account of its great convenience, being arranged like logarithms.

1871 Schott. 'Method of adjustment of the secondary triangulation of Long Island Sound.' *Rep. Coast Sur. U. S.* for 1868, pp. 140–146.

1871 Zachariae. '*De mindste Qvadraters Methode.*' Nyborg, 8vo, pp. viii, 234.

This is an excellent text-book. See review in *Jahrb. Fortschr. Math.*, Vol. III, p. 95.

1872 Dienger. 'Ueber einen Satz der Wahrscheinlichkeitsrechnung und damit zusammenhängende bestimmte Integrale.' *Abhandl. Böhmis. Gesell.*, Vol. V, 44 pp.

See *Jahrb. Fortschr. Math.*, Vol. IV, p. 89.

1872 Glaisher. 'Remarks on certain portions of Laplace's Proof of the Method of Least Squares.' *Lond. Phil. Mag.*, Vol. XLIII, pp. 194–201.

The matter of this paper is mostly included in the following.

1872 Glaisher. 'On the Law of Facility of Errors of Observations, and on the Method of Least Squares.' *Mem. Astron. Soc. Lond.*, Vol. XXXIX, pp. 75–124.

This is perhaps the most valuable of all the theoretical memoirs on our list, presenting as it does clear critical analyses of the principal proofs of the law $\varphi(x)=ce^{-h^2x^2}$ and of the Method of Least Squares. It has been of great value to me in preparing this list.

ADRAIN's first proof is examined at length and its reasoning shown to be defective. Then are analysed in order: 1. GAUSS's first proof, including ENCKE's, DEMORGAN's and ELLIS's remarks on the arithmetical mean; 2. LAPLACE's method, POISSON's and ELLIS's simplifications and IVORY's criticisms; 3. GAUSS's second demonstration; 4. HERSCHEL's proof, with ELLIS's and BOOLE's criticisms thereon; 5. TAIT's and similar proofs; 6. DONKIN's proof of 1857. By means of the index at the end of this list the reader may refer back to these papers, where I have often quoted GLAISHER's remarks.

It is considered unproved that the arithmetical mean gives the most probable result. GAUSS's second proof is regarded as resting upon an arbitrary assumption, which practically assumes the point to be proved. LAPLACE's method is considered as giving the only correct and philosophical analysis of the question, and this GLAISHER shows leads directly to the exponential law of facility, provided that the sources of error are very great in number and that positive and negative errors are equally likely. "TAIT's proof" is found insufficient. The proofs of 1837 HAGEN, 1838 BESSEL, 1844 DONKIN and 1870 CROFTON are not discussed.

PEIRCE's criterion for the rejection of doubtful observations is regarded as "destitute of scientific precision." ".... under no circumstances have we a right to say an observation has no weight, though it may be better to give it none than to give it as much as the best." The method of assigning weights in such cases is hinted at; see below 1873 GLAISHER.

For accounts of the contents of the memoir see *Monthly Notices*, Vol. XXXII, p. 241, and *Jahrb. Fortschr. Math.*, Vol. IV, p. 92.

1872 HELMERT. '*Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate mit Anwendungen auf die Geodäsie und die Theorie der Messinstrumente*,' Leipzig, 8vo, pp. xi, 348.

The exponential law is regarded as a law proved by experience. The arithmetical mean is said to be the *most plausible* value. Both the first and second proofs of GAUSS are given, and the second is regarded as better and more general.

While the theoretical part of the book is not satisfactory, the practical part renders it valuable for geodetic engineers. Conditioned observations in particular are well treated.

1872 HILGARD. 'An application of an Exponential Function.' *Proc. Amer. Assoc.* for 1871, pp. 61–63.

A certain statute relating to errors in coinage is discussed.

1872 HOPKINSON. On the calculation of empirical formulæ. *Messenger Math.*, Vol. II, pp. 65–67.

A method less accurate than Least Squares.

1872 JORDAN. 'Vergleichung der Genauigkeit verschiedener Gradmessungen.' *Astron. Nachr.*, Vol. LXXX, col. 17–22.

The mean errors of 17 angle measurements and 21 base line measurements are given, the latter for a line one kilometer in length. The greatest mean error of a base line measurement is 63.2 mm. and the least 0.12 mm., the first being measured in 1739 and the second in 1860. This is one of those papers in which the results of long continued research and labor are expressed in a few lines.

1872 JORDAN. 'Ueber die Bestimmung des mittleren Fehlers durch Wiederholung der Beobachtungen.' *Astron. Nachr.*, Vol. LXXIX, col. 219–222.

On the method of deducing probable errors from the $\frac{1}{2}n(n-1)$ differences of n measurements, given by him in 1872. See next article.

1872 ANDRÄ. 'Ueber die Bestimmung des wahrscheinlichen Fehlers durch die gegebenen Differenzen von m gleich genauen Beobachtungen einer Unbekannten.' *Astron. Nachr.*, Vol. LXXIX, col. 257–272.

An investigation of JORDAN's method leading to somewhat different results. See *Jahrb. Fortschr. Math.*, Vol. IV, p. 577, and *Bull. Math. et Astron.*, Vol. IX, pp. 27. See below 1876 HELMERT.

1872 JORDAN. 'Ueber die Bestimmung des Gewichts einer durch die Methode der kleinsten Quadrate bestimmten Unbekannten.' *Zeitschr. Math. u. Phys.*, Vol. XVII, pp. 350–352.

A simplification of demonstrations of two methods.

1872 LORENZ. Udjevning af Jagttagelses fyl. *Tidsskr. for Math.*, Vol. II, pp. 1– . . , 97, 125, 162.

See *Jahrb. Fortschr. Math.*, Vol. IV, p. 94. The same Journal contains articles by ZACHARIAE.

1872 RUMPEN. '*Ueber den Zusammenhang der von* GAUSS *begründeten Methode der kleinsten Quadrate mit der algebraischen Theorie der quadratischen Formen.*' Bonn, 1872, 8vo, pp. 40.

A doctor's thesis. The conditions for minimum squares, etc., discussed by help of determinants.

1872 SCHLÖMILCH. 'Ueber die Bestimmung der Wahrscheinlichkeit eines Beobachtungsfehlers.' *Zeitschr. Math. u. Phys.*, Vol. XVII, pp. 87–88.

HERSCHEL's proof is taken from 1869 THOMPSON and TAIT, and pronounced "einfache und anschauliche."

1872 Woolhouse. 'On the Philosophy of Statistics.' *Comp. to Almanac* for 1872, pp. 5–22. —*Lond. Assur. Mag.*, XVII, pp. 26–30.

A popular article copied largely from Quetelet.

1872 Zachariae. 'Note betreffend die Bestimmung des mittleren Fehlers.' *Astron. Nachr.*, Vol. LXXX, col. 67–70.

Offers without demonstration a new formula for the mean error of a base line measured in several portions. See 1873 Helmert and Jordan.

1873 Brown. 'On the Application of the Binomial Law to Statistical Enquiries, illustrated by the Law of the Growth of Man at different ages.' *Lond. Assur. Mag.*, Vol. XVII, pp. 340–351.

1873 Crookes. 'On the Probability of Error in Experimental Research.' *Quart. Jour. Sci.*, Vol. III, pp. 1–13.

Contains some formulæ for weights taken from 1838 DeMorgan, with an application to determining the probability of error in the atomic weight of thallium.

1873 DeForest. 'On some Methods of Interpolation applicable to the graduation of irregular Series, such as Tables of Mortality, &c. &c.' *Rep. Smithsonian Inst.* for 1871, pp. 273–339; for 1873, pp. 319–355.

For continuation see below under 1876.

1873 Helmert. 'Bestimmung des mittlern Fehlers der Längen messungen aus der Differenzen von Doppelmessungen.' *Astron. Nachr.*, Vol. LXXXI, col. 49–52.

A comparison of the common formulæ with the one given by Zachariae in 1872, showing that the latter is less accurate.

1873 Jordan. 'Ueber die Berechnung der mittleren Fehlers einer Basismessung.' *Astron. Nachr.*, Vol. LXXXI, col. 51–56.

Also a criticism on Zachariae's formula.

1873 Zachariae. 'Ueber die Bestimmung des mittleren Fehlers einer in meheren Theilen doppelt gemessenen Grundlinie.' *Astron. Nachr.*, Vol. LXXXI, col. 225–228.

Defends his formula against the above. See *Jahrb. Fortschr. Math.*, Vol. V, p. 127.

1873 Jahn. *Sur la probabilité du tir et la méthode des moindres carrés.* Paris, 8vo.

1873 JORDAN. 'Verallgemeinerung eines Satzes der Methode der kleinsten Quadrate.' *Zeitschr. Math. u. Phys.*, Vol. XVIII, pp. 116–120.

Concerning the solution of normal equations.

1873 JOUFRET. *Sur la methode des moindres carrés et ses applications du tir.* Paris, 8vo.

1873 LAURENT. '*Traité du calcul des probabilités.*' Paris, 8vo. pp. xii, 268.

This is intended as an introduction to the study of LAPLACE's *Théorie.... des Prob.* At the end is the best list of literature on the Method of Least Squares which I have seen.

1873 NEWCOMB. 'A mechanical Representation of a familiar Problem.' *Monthly Notices Astron. Soc. Lond.*, Vol. XXXIII, pp. 573–574.

"Given at several epochs, observed values of a quantity which varies uniformly with the time, to find by Least Squares the most probable values of the two constants which fix its value at any time." The analogy of the question with one of equilibrium in mechanics is pointed out. The solution of a system of linear equations by Least Squares may be represented in a similar way.

1873 PEIRCE (C. S.) 'On the theory of errors of observations.' *Rep. Coast Sur. U. S.* for 1870, pp. 200–224.

Gives an account of some interesting experiments "made to study the distribution of errors in the observation of a phenomena not seen coming on, as in the case of a transit, but sudden as in the case of the emersion of a star from behind the moon." The results are given graphically and show a decided approximation to the exponential law of facility.

In the ten pages of introduction new ideas are offered concerning a notation, "suggested by the study of the logic of relations."

1873 SEELIGER. 'Ueber die JACOBI'sche Auflösung eines Systems von Normalgleichungen mit drei Unbekannten.' *Astron. Nachr.*, Vol. LXXXII, col. 249–252.

See above 1840 BESSEL.

1873 STONE. 'On the most Probable Result which can be derived from a number of direct Determinations of Assumed Equal Values.' *Month. Not. Astron. Soc. Lond.*, Vol. XXXIII, pp. 570–572.

Shows that the arithmetical mean is the most probable results for $n+1$ observations, provided it is the most probable result for n observations, and as it is undoubtedly such for $n=2$, " . . . it can be shown to be generally true."

1873 WREDE. 'Några anmärkningar rörande minste qvradtmethoden.' *Öfversigt Förhandl. Acad. Stockholm*, Vol. XXX, No. 8, pp. 3–34; No. 10, pp. 21–26.

The probable error is said to be not always 0.6745 of the mean error but depends upon the number of unknown quantities involved. See 1852 BIENAYMÉ. BESSEL's investigation of 1838 is also discussed.

1873 GLAISHER. 'On the Rejection of Discordant Observations.' *Monthly Notices Astron. Soc. Lond.*, Vol. XXXIII, pp. 391–402.

It is here clearly pointed out how inconsistent is the rejection of discordant observations by a criterion founded on the supposition of the validity of the arithmetical mean. The idea first advanced by DEMORGAN (*Encyc. Metrop.*, 1847) that the mean is only an approximate value to be used in weighting the observations from which a new mean is to be deduced, and so on, is here developed to a certain extent. See 1821 ——. The criterion given by STONE in 1868 is examined and pronounced untrustworthy and wrong.

1873 STONE. 'On the Rejection of Discordant Observations.' *Monthly Notices*, Vol. XXXIV, pp. 9–15.

A reply to the preceding in which GLAISHER's arguments are examined at length and the validity of the criterion maintained. GLAISHER's method for weighting observations is also discussed and regarded "as mathematically unsound."

1874 GLAISHER. 'Note on a paper by Mr. STONE, "On the Rejection of Discordant Observations."' *Monthly Notices*, Vol. XXXIV, p. 251.

1874 STONE. 'Note on a Discussion relating to the Rejection of Discordant Observations.' *Monthly Not.*, Vol. XXXV, pp. 107–108.

1874 CANTOR. '*Historische Notizen über die Wahrscheinlichkeitsrechnung.*' Halle, 8vo, pp. 8.

This is of no value. GAUSS alone is mentioned in connection with Least Squares.

1874 FECHNER. 'Ueber die Bestimmung des wahrscheinlichen Fehlers eines Beobachtungsmittels durch die Summe der einfachen Abweichungen.' *Poggendorff's Annal. Phys.*, Jubelband, pp. 66–81.

This is an extract from a memoir published in *Abhandl. Sächs. Gesell;* see below 1875. FECHNER deduces the formula

$$r_0 = \frac{1.195502}{\sqrt{2n - 0.8548}} \cdot \frac{\Sigma x}{n}$$

n being the number of observations and Σx the sum of the residuals all taken positively. See 1876 HELMERT.

1874 GLAISHER. 'On the Solution of the Equations in the Method of Least Squares.' *Month. Not. Astron. Soc. Lond.*, Vol. XXXIV, pp. 311–334.

A critical discussion of GAUSS's, BESSEL's and JACOBI's methods for solving equations and determining weights.—See *Jahrb. Fortschr. Math.*, Vol. VI, p. 145.

1874 HANSEN. 'Von der Bestimmung der Theilungsfehler eines gradlinigen Maassstabes.' *Abhandl. Sächs. Gesell.*, Vol. XV, pp. 525–668.

1874 JEVONS. '*The Principles of Science: a Treatise on Logic and Scientific Method.*' London, 8vo. Vol. I, pp. xvi, 463; Vol. II, pp. vii, 480.

Chapters on 'The Method of Means,' 'The Law of Error,' 'Theory of Approximation,' etc., give valuable remarks concerning the fundamental principles of the Method of Least Squares.

1874 JORDAN. *Taschenbuch der praktischen Geometrie. Eine Sammlung von Resultaten der höheren und niederen Vermessungskunde.* Stuttgart, 8vo, pp. xi, 416.

In Part I the Method of Least Squares is presented. See review in *Schlömilch's Lit. Zeit.*, Vol. XVIII, pp. 33–40. The book is now (1877) being republished under the title '*Elemente der Vermessungskunde;*' and Part 1, pp. 1–136, contains a valuable treatise on Least Squares.

1874 MEYER. 'Calcul des Probabilités de A. MEYER, publie sur les manuscrits de l'auteur par F. FOLIE.' *Mém. Soc. Liége*, Vol. VI, No. 2, x+446 pp. —Also published under title *Cours de Calcul.... fait à la Universite de Liége....* Bruxelles, 8vo.

This is a comprehensive and valuable work. In the parts relating to the theory of observations the proofs of 1809 GAUSS and 1837 HAGEN are given, as also the investigations of 1852 BIENAYMÉ.

1874 POWALKY. 'On the Combination of the different Results of various Series of Observations.' *Month. Not. Astron. Soc. Lond.*, Vol. XXXIV, pp. 476–479.

1874 SEIDEL. 'Ueber die Berechnung der wahrscheinlichsten Werthe solcher Unbekannten zwischen welchen Bedingungs-Gleichung bestehen.' *Astron. Nachr.*, Vol. LXXXIV, col. 193–210.

The conditional equations are regarded as having infinite weights. This way of consideration appears to lead to a new method of solution.

1874 SEIDEL. Ueber ein Verfahren, die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, so wie lineäre Gleichungen überhaupt, durch successive Annährung aufzulösen. *Abhandl. Akad. München* for 1874, pp. ....

See *Jahrb. Fortschr. Math.*, Vol. VI, p. 147.

1875 AIRY (W.). 'On the Probable Errors of Levelling; with Rules for the Treatment of Accumulated Errors.' *Proc. Inst. Civ. Engs.* for 1875, pp. .... —*Engineering News*, Vol. IV, pp. 77–78, 84–85, 92, 98–99, 104–105.

A very valuable practical paper.

1875 BAEYER. 'Ueber Fehlerbestimmung und Ausgleichung eines geometrischen Nivellements.' *Astron. Nachr.*, Vol. LXXXVI, col. 177–188.

BESSEL's method for adjusting a triangulation is applied to a connected system of levels.

1875 BELLATI. Intorno ad un modo di semplificare in alcuni casi l'applicazione del metodo dei minimi quadrati al calcolo delle costanti empiriche. *Atti Istit. Veneto*, Vol. I.

1865 BIENAYMÉ. 'Application d'un théorème nouveau du Calcul des probabilités.' *Comptes Rendus Acad. Paris*, Vol. LXXXI, pp. 417–423. —*Bull. Math. et Astron.*, Vol. IX, pp. 219–225.

If a series of observations be arranged in the order of the measurements, there are certain maxima and minima whose probable number and position are given by the theorem. On pp. 458–459, 491–492 of this volume of the *Comptes Rendus* are remarks by BERTRAND on the theorem.

1875 DIENGER. 'Die LAPLACE'sche Methode der Ausgleichung von Beobachtungsfehler bei zahlreichen Beobactungen.' *Denkschr. Akad. Wien*, Vol. XXXIV, 42 pp.

The method is extended to the case of several unknown quantities. See *Repertorium Math.*, Vol. I, p. 241.

1875 Faye. 'Note accompagnant la presentation d'une Notice autographiée sur la méthode des moindres carrés.' *Comptes Rendus, Paris.*, Vol. LXXX, pp. 352–356.

The exponential law of error is regarded as an empirical law established by experience.

[1875] Fechner. Ueber den Ausgangswerth der kleinsten Abweichungssumme, dessen Bestimmung und Verwendung. *Abhandl. Sächs. Gesell.*, Vol. [XVI].

See title 1874 Fechner.

1875 Franke. 'Die trigonometrische Punktbestimmung im Netz-Auschluss, mit besonderer Rücksicht auf eine rationelle Fehler-Ausgleichung.' München, 8vo, pp. viii, 69.

Reference is here made to articles by Tulla, Jordan and others on a graphical method of adjustment, whose titles I regret not to be able to give. See *Monatsbl. Badisch. Geometervereins* for 1875.

1875 Galton. 'Statistics by Intercomparison, with Remarks on the Law of Frequency of Error.' *Lond. Phil. Mag.*, Vol. XLIX, pp. 33–46.

If all the men of a tribe were arranged in a row according to their heights, the middle man would have the *mean* height.

The curve $y = ce^{-h^2x^2}$ is called an "ogive" and it is regarded as more likely to be approximately true of a statistical series than any other that can be specified *à priori.*

1875 Helmert. 'Ueber die Formel für den Durchschnittsfehler.' *Astron. Nachr.*, Vol. LXXXV, col. 353–366.

The formula given in 1856 by Peters is discussed, and shown to be correct only for direct observations. A new formula for probable error is proposed. See 1869 Lüroth and 1876 Helmert.

1875 Laurent. 'Sur la méthode des moindres carrés.' *Liouville's Jour. Math.*, Vol. I, pp. 75–80.

A discussion of 1444 observations to deduce an empirical law of error. The result is that the exponential law represents closely the probabilities of error.

1875 Mees. Ueber die Berechnung des wahrscheinlichen Fehlers einer endlichen Zahl von Beobachtungen. *Zeitschr. Math. u. Phys.*, XX, pp. 145–152.

Gauss's method (1816) is considered incorrect.

1875 HELMERT. Ueber die Berechnung des wahrscheinlichen Fehlers aus einer endlichen Anzahl wahrer Beobachtungsfehler. *Zeitschr. Math. u. Phys.*, Vol. XX, pp. 300–303.

A reply to MEES's article above.

1875 NATANI. '*Methode der kleinsten Quadrate. Mit den Hülfsatzen aus der Analysis und Wahrscheinlichkeitsrechnung nebst einem Anhange über die ballistische Linie.*' Berlin, 8vo, pp. 42.

HERSCHEL's proof is given. The numerical examples concern the probability of striking a target.

1875 SCHIAPARELLI. 'Sur le principe de la moyenne arithmétique.' *Astron. Nachr.*, Vol. LXXXVII, col. 55–58.

A demonstration that the arithmetical mean of direct observations gives "le seul résultat plausible et conciliable avec les exigences pratiques de la question." See 1848 MATZKA, 1868 SCHIAPARELLI and 1876 STONE.

1875 SCHOLS. Over de theorie der fouten in de ruimte en in het platte vlak. *Verhandl. Akad. Amster.*, Vol. XV.

See *Jahrb. Fortschr. Math.*, Vol. VII, p. 114.

1875 SCHOLS. De interpolatie-formule von TCHÉBYCHEF volgens de methode der kleinste vierkanten. *Versl. Akad. Amsterdam*, Vol. IX, pp. 301–311.

1875 SCHOTT. 'Determination of weights to be given to observations for determining time with portable transit instruments, recorded by the chronographic method.' *Rep. Coast Sur. U. S.* for 1872, pp. 222–226.

1875 TILLY. Note sur le principe de la moyenne arithmetique et sur son application à la théorie mathématique des erreurs. *Nouv. Corres. Math.*, Vol. I, pp. 137–147.

See *Bull. Math. et Astron.*, Vol. XII, p. 146.

1875 TILLY. Théorie mathématique des erreurs. *Ballistique*, (Bruxelles, 8vo), pp. 155–225.

The methods of GAUSS, HAGEN and LAPLACE compared. See review in *Jahrb. Fortschr. Math.*, Vol. VII, p. 108.

1875 VAN GEER. Over het gebruick von determinanter by de methode der kleinste kwadrater. *Nieuw Arch.*, Vol. I, pp. 179–188.

1876 BAUERNFEIND. 'Methode der kleinsten Quadrate.' *Elemente der Vermessungskunde* (Stuttgart, fifth edition, 2 vols., 8vo), Vol. II, pp. 4–30.

An elementary sketch of the method.

1876 CHAMBERS (C.) and CHAMBERS (F.). 'On the Mathematical Expression of Observations of Complex Periodical Phenomena; and on Planetary Influence on the Earth's Magnetism.' *Phil. Trans. Lond.*, Vol. CLXV, pp. 361–402.

1876 DEFOREST. '*Interpolation and Adjustment of Series.*' New Haven, 8vo, pp. 52.

A supplement to his memoirs of 1873. Besides other valuable matter, methods for finding probable errors of adjusted terms are given.

1876 FERRERO. *Esposizione del metodo dei minimi quadrati.* Firenze, 8vo, pp. 234.

1876 HAGEN. '*Untersuchungen über die gleichförmige Bewegung des Wassers.*' Berlin, 8vo, pp. 104.

All known observations on the mean velocity of rivers are discussed by the Method of Least Squares, and the most probable law and formula for mean velocity are deduced.

1876 HELMERT. 'Die Genauigkeit der Formel von PETERS zur Berechnung des wahrscheinlichen Fehlers directer Beobachtungen gleicher Genauigkeit.' *Astron. Nachr.*, Vol. LXXXVIII, col. 113–132.

Simplifications are given of HELMERT's formula of 1875, and the formulæ of FECHNER, JORDAN and ANDRÄ (see 1869–1874) are discussed.

1867 HELMERT. Ueber die Wahrscheinlichkeit der Potenzsummen der Beobachtungsfehler und über einige damit im Zusammenhang stehende Frage. *Zeitschr. Math. u. Phys.*, Vol. XX, pp. 192–218.

1876 KUMMELL. 'New Investigation of the Law of Errors of Observation.' *The Analyst*, Vol. III, pp. 133–140, 165–171.

HAGEN's proof of 1837 is given abbreviated and improved, and the usual rules for normal equations and probable errors are deduced. The probability to commit no error at all is regarded as an absolute constant. See *Jour. Franklin Inst.*, 1877, Vol. CIV, pp. 270–274.

1876 LÜROTH. 'Vergleichung von zwei Werthen des wahrscheinlichen Fehlers.' *Astron. Nachr.*, Vol. LXXXVII, col. 209–220.

The usual formula is compared with a new formula and shown to give larger values.

1876 SAFFORD. 'On the Method of Least Squares.' *Proc. Amer. Acad.*, Vol. XI, pp. 193–201.

"The main object of this paper is to give rules for good observing derived from this theory." Hints for abbreviating computations are added.

1876 SKINNER. '*Principles of Approximate Computations.*' New York, 12mo, pp. v, 98.

Presents simple rules for conducting computations involving approximate quantities, in such a manner as to require the fewest figures and to show at once the degree of accuracy of the result.

1876 STONE. 'Sur le principe de la Moyenne arithmétique. *Astron. Nachr.*, LXXXVIII, col. 61–64.

Points out that some of the assumptions of SCHIAPARELLI's proof of 1875 agree with those of his own proof of 1873. The article is in English.

1876 VENN. '*The Logic of Chance. An Essay on the foundations and province of the Theory of Probability, with special reference to its logical bearings and its applications in Moral and Mental Science.*' London, second edition, 8vo, pp. xxvii, 488.

VENN's views are: First, almost any regular and symmetrical method of treating the errors of observation will tend to approximate indefinitely toward the truth as the number of observations is indefinitely multiplied, and this whatever be the law of facility; secondly, the Method of Least Squares is the best method (upon the reasonably probable supposition of the universality of the exponential law), that is, it approximates quicker to the truth as the number of observations is increased than any other method; but its superiority over other reasonable methods is small in comparison with their common superiority over single observations.

'*Jahrbuch über die Fortschritte der Mathematik.*' Berlin, 8vo. One vol. of about 750 pages appears yearly in 3 parts.

This invaluable publication has been of great use to me in preparing the above list for the years 1868–75. Vol. VIII (not yet issued) embracing the literature for 1876, will undoubtedly contain the titles of some writings on the Method of Least Squares which are not given here.

INDEX OF AUTHORS.

INDEX OF SUBJECTS.

ERRATA.

Page 172, line 21 from foot, *for* Remarks *read* 'Remarks.
" 173, " 11 " top, " Jái " J'ai.
" 179, " 8 " foot, " unhekannten *read* unbekannten.

ADDENDA.

1876 RÜDIGER. *Die Methode der kleinsten Quadrate abgeleitet aus der Wahrscheinlichkeitslehre, und ihre Anwendung auf naturwissenschaftlichen Messungen.* Frankfurt an der Oder, 8vo, pp. 48.

A doctor's dissertation.

1876 ———. '*Zusammenstellung der Literatur der Gradmessungs-Arbeiten.*' Berlin, 4to, pp. 32.

This is drawn up by commissioners of the states and countries belonging to the European International Geodetic Survey. It contains references to about 380 writings on Geodesy, 200 of which are German, 64 English, 50 Italian and 24 French. The English and Italian literature is well presented, the German and French is not. Coming from such a source, this list should have been a great deal better.

The work is received just as this sheet goes to press, and the following are the additions which it renders necessary to the preceding pages:

The work by BAEYER recorded on page 206 is a lithographed manuscript issued in 1867 or 1868. Two other parts on Geodesy were also published.

The following articles, all relating to the adjustment of geodetic triangulations, should properly have been mentioned in my list: 1865 SCHIAPARELLI, *Istit Lombard. Rend.*, Vol. II, pp. 348–359; Vol. III, pp. 27–41. 1867 WITTSTEIN, *Astron. Nachr.*, Vol. LXIX, col. 289–298; 321–328. 1868 PRONDZYNSKI and BÖRSCH, *Astron. Nachr.*, Vol. LXXI, col. 145–154; 265–268; Vol. LXXV, col. 87–90.

A lithographed manuscript by BAEYER, entitled *Untersuchungen über die Ausgleichung nach Winkel- und Seitengleichungen*, was published in 1871.