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A. de Moivre: 'De Mensura Sortis' or 'On the Measurement of Chance'

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Commentary on 'De Mensura Sortis'

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Summary

De Moivre's first work on probability, *De Mensura Sortis seu, de Probabilitate Eventuum in Ludis a Casu Fortuito Pendentibus*, is translated into English. A commentary gives a sketch of the historical background, a summary of the contents of *De Mensura Sortis* in modern notation and terminology and some remarks on the priority dispute between Montmort and de Moivre.

Key words: De Mensura Sortis; De Moivre; History of probability; Montmort; Priority dispute; Translation.

1 The historical background

It is generally agreed that probability theory was born in 1654 by the correspondence between Pascal and Fermat on the division problem (the problem of points) even if a beginning had been made previously by Cardano and Galileo. The Pascal–Fermat letters were only published much later, but as usual at this time their contents were communicated to a number of colleagues. However, an essential part of Pascal's contribution is also included in his *Traité du Triangle arithmétique* published in 1665. Pascal and Fermat used the addition theorem and the multiplication theorem for independent events without comments as if these theorems were generally known. They solved the division problem in two ways: by combinatorial methods and by a recursive method (a difference equation). Their results were expressed by means of the negative binomial distribution and the binomial distribution for $p = \frac{1}{2}$.

During a visit to Paris in 1655 Huygens heard about these problems and after his return to the Netherlands he wrote his treatise *De Ratiociniis in Aleae Ludo* which was published in 1657 and for the next 50 years was the only treatise on probability theory. Huygens solved the division problem and many other problems by recursion; he did not use combinatorial methods. At the end of his treatise he formulated five problems as a challenge to other mathematicians. Solutions, interpretations and generalizations of these problems were discussed by Huygens himself and by Hudde, Spinoza, Bernoulli, Montmort, de Moivre and Struyck.

It is a peculiar fact that no essential contribution to probability theory was published between 1657 and the publication in 1708 of Montmort's book *Essay d'Analyse sur les*

Jeux de Hazard. The reason for this may perhaps be that the mathematical world was fully occupied by developing the calculus. Furthermore, probability theory had not yet got any important application in science, but only in life insurance.

The real breakthrough in probability theory took place in 1708–1713 with the publication of Montmort's book in 1708, de Moivre's treatise *De Mensura Sortis* in 1712, Bernoulli's book *Ars Conjectandi* in 1713 and finally the second and much enlarged edition of Montmort's book in 1713. To a large extent these works treat the same problems, namely the problems of finding the probability of winning or the expected gain in various games of chance. The notable exception is Bernoulli's proof of the law of large numbers for a binomially distributed random variable which established the first limit theorem in probability theory. Besides the combinatorial and recursive methods used by previous authors they used conditional probabilities and expectations, infinite series and the method of inclusion and exclusion.

The assignment of priorities to the many new discoveries is rather involved. When James Bernoulli died in 1705 he left a nearly complete manuscript to his book, but due to quarrels within the Bernoulli family it was not published until 1713; see Kohli (1975a). Hence, Bernoulli's results existed before Montmort's and de Moivre's but they were published later. Montmort wrote his book independently of Bernoulli and he was thus the first to publish many new results. Todhunter (1865, p. 76) writes about Montmort:

> In 1708 he published his work on Chances, where with the courage of Columbus he revealed a new world to mathematicians.

Montmort's book is an impressive work, but it is also in many places discursive and obscure. This is no wonder since the subject was new and difficult. Many of the games analysed were extremely complicated and the solutions required great combinatorial skill. De Moivre (1718, p. 70) writes about Montmort's solution of Robartes' problem:

> I very willingly acknowledge his Solution to be extreamly good, and own that he has in this, as well as in a great many other things, shewn himself entirely master of the doctrine of Combinations, which he has employed with very great Industry and Sagacity.

Undoubtedly Montmort felt that many of the problems could be pursued further and also that many further problems should be taken up and he began his work on a revised, much improved and enlarged second edition in co-operation with Nicholas Bernoulli, a nephew of James and John Bernoulli. The second edition containing letters between John and Nicholas Bernoulli and Montmort was published in 1713.

De Moivre worked on probability theory from 1708. Since he based his work on Huygens' treatise and Montmort's book it is quite natural that his results are rather similar to the results found by Montmort and Nicholas Bernoulli. On de Moivre's paper Todhunter (1865, p. 140) writes:

> Many important results were here first *published* by de Moivre, although it is true that these results already existed in manuscript in the *Ars Conjectandi* and the correspondence between Montmort and the Bernoullis.

2 A summary of the contents of *De Mensura Sortis*

De Mensura Sortis takes up No. 329 of the *Philosophical Transactions* for the months January, February and March, 1711, and is a part of Vol. 27 published in 1712. Nearly all of *De Mensura Sortis* was later incorporated into de Moivre's book *The Doctrine of Chances* (1718, 1738, 1756), which was the most important textbook on probability theory until the publication of Laplace's *Théorie Analytique des Probabilités* (1812).

In the preface de Moivre states that he began his work on probability theory at the exhortation of Francis Robartes, who asked him to solve the division problem for two gamesters playing bowls and also to find the probability of getting certain given faces as the outcome of a given number of throws with a die. He also states that he had previously read the books by Huygens and Montmort 'but these distinguished gentlemen do not seem to have employed that simplicity and generality which the nature of the matter demands'. Furthermore he writes that 'while they suppose that the skill of the gamesters is always equal, they confine this doctrine of games within limits too narrow'. Finally his remarks about Montmort may be read as if Montmort had used only the method of Huygens on some new examples. These rash remarks naturally provoked a dispute with Montmort.

De Moivre writes in an elaborate style. Usually he begins with a simple numerical example before stating the general result. Often the proof is indicated only in the example and in some cases no proof is given. All the 26 problems discussed are games of chance. No observations (relative frequencies) are mentioned. Most of the problems are formulated in terms of odds or expectations (the stake being 1) so that the word 'probability' is not used as much as later became customary.

We shall give a summary of the contents of *De Mensura Sortis* using modern notation and terminology and classifying the 26 problems under the main headings which would have been used today.

The multiplication theorem. This theorem is stated clearly for independent events in the introduction. However, many problems lead to 'consecutive drawings without replacement' and they are solved by means of the multiplication theorem for dependent events.

The binomial distribution and the approximating Poisson distribution. The binomial distribution is derived in the introduction and used to find the odds for a player contending that he will win at least c times in n trials; an example is given in Problem 1. In Problems 5–7 de Moivre generalizes the old problem of finding the number of trials that gives an even chance for getting at least one success to any given number of successes, c say. This means that he has to solve the equation $B(c-1, n, p)=\frac{1}{2}$ with respect to n for given c and p, where B denotes the binomial distribution function. De Moivre considers the two extreme cases $p=\frac{1}{2}$ and $p\to 0$. For $p=\frac{1}{2}$ the solution is $n=2c-1$ because of symmetry. For $p\to 0$ and $n\to\infty$ de Moivre proves that

$$\sum_{x=0}^{c-1}\binom{n}{x}p^x q^{n-x}\to e^{-m}\sum_{x=0}^{c-1} m^x/x!, \quad m=np/q.$$

However, he was not able to write his result in the form given above because the notation for the exponential function had not yet been invented. Instead of e^m he had to write 'the number of which the hyperbolic logarithm is m'. He therefore wrote his result in logarithmic form as

$$m=\log 2+\log\{1+m+m^2/2!+\ldots+m^{c-1}/(c-1)!\},$$

which also is natural for computational purposes. He solves this equation for $c=1,\ldots,6$ and states that $m\sim c-\frac{1}{2}$ for $c\to\infty$. (As is well known today the limiting value is $c-\frac{1}{3}$.)

The importance of de Moivre's result was recognized by Simpson (1740, pp. 33–36) who extended de Moivre's table to $c=10$. It is peculiar that Laplace and Poisson do not refer to de Moivre's solution.

There has been some discussion of whether it is reasonable to contend that de Moivre found the Poisson distribution, the latest contribution being that of Stigler (1982).

The hypergeometric distribution, Huygens' fourth problem. In Problem 14 the hypergeometric distribution is derived by the usual combinatorial method.

The division problem. Consider a series of games with two players, A and B, where in each game A has probability p and B probability $q=1-p$ of winning a point. If they stop playing when A lacks a points and B lacks b points in winning, how should the stake be divided between them? De Moivre proves that A's probability of winning equals the sum of the last b terms of the expansion of $(p+q)^{a+b-1}$, and B's probability of winning equals the remaining a terms; see Problems 2, 3, 4 and 10.

This result had already been derived by John Bernoulli in 1710 in a letter to Montmort, but it was not published until 1713; see Montmort (1713, p. 295). Montmort (1713, pp. 245–246) also derived the solution in terms of the negative binomial distribution and showed that the two formulae gave the same result, thus generalizing the two results of Pascal and Fermat mentioned above.

In Problem 8 de Moivre generalizes to k players, say, and gives the solution as the sum of the appropriate terms of the multinomial $(p_1+p_2+\ldots+p_k)^{n+1-k}$, n being the total number of points lacking. He points out that certain terms have to be divided among the players depending on the permutation of the p's.

In Problems 16 and 17 he gives the solution of Robartes' problem: the division problem for two gamesters playing bowls. In each game B, say, gets a number of points equal to the number of his bowls which is nearer to the jack than any of A's bowls. By combinatorial methods de Moivre finds the probability of getting i points in a single game assuming that the players have the same number of bowls and are of the same skill. The division problem is then solved by recursion.

Huygens' first and second problem. In these problems the players take turns in a specified order until one of them wins; see Problems 11, 12 and 13. De Moivre gives the solution as the sum of an infinite series.

Waldegrave's problem; *Problem* 15. De Moivre solves this problem by means of conditional expectations. First he supposes that A beats B in the first game. On this assumption the play may end with A as winner in the second, fifth, eighth, ... game. The probabilities of A for reaching these games and winning are $\frac{1}{2}$, $(\frac{1}{2})^4$, $(\frac{1}{2})^7$, ... Hence the expected stake plus fines may be found and subtracting A's expected fine his (conditional) expectation results. Under the same assumption B's expectation is obtained. The unconditional expectation is then found as the average of the two conditional expectations.

The occupancy problem; *Problems* 18 *and* 19. Let k be the number of faces on a die (the number of cells) and let n be the number of trials (the number of balls to be distributed at random in the cells). The probability that for a given set of f faces all the faces occur at least once in n trials equals

$$P_n = \sum_{i=0}^{f} (-1)^i \binom{f}{i}\left(1-\frac{i}{k}\right)^n.$$

This result is proved by de Moivre in Problem 18 by means of the method of inclusion and exclusion.

In Problem 19 he solves the equation $P_n = \frac{1}{2}$ under the assumption that f is small compared to k. Setting $1-(i/k) \sim 1/r^i$ for $1/r = (k-1)/k$, he finds $P_n \sim (1-r^{-n})^f = \frac{1}{2}$, which is easily solved for n.

The ruin problem, Huygens' fifth problem; *Problem* 9. De Moivre generalizes Huygens' problem as follows. Consider two players A and B having a and b counters, respectively. In each game A has probability p and B has probability $q=1-p$ of winning and the winner gets a counter from the loser. The play continues until one of the players is ruined. What is the probability of A being ruined? Huygens' fifth problem is obtained for $a=b=12$.

De Moivre's proof depends on an ingenious trick. Suppose that A's counters are numbered from 1 to a and B's from $a+1$ to $a+b$ and that counter no. x is given the fictitious value $(q/p)^x$. Suppose further that in each game only consecutively numbered counters are used, such that in the first game A's stake is $(q/p)^a$ and B's is $(q/p)^{a+1}$. If A wins he gets counter number $a+1$ from B and in the next game A's stake is therefore $(q/p)^{a+1}$ and B's is $(q/p)^{a+2}$ and so on. Hence, in any trial A's expectation will be $p(q/p)^{x+1}-q(q/p)^x=0$ and B's expectation will similarly be 0. Since they have the same expectation in every trial their total expectations must also be the same. However, the total expectation equals the probability of winning times the total amount won at the end of the play, so that

$$P_a\{(q/p)^{a+1}+\ldots+(q/p)^{a+b}\}=P_b^*\{(q/p)+\ldots+(q/p)^a\},$$

P_a and P_b^* denoting the probabilities of winning. Assuming that $P_a+P_b^*=1$ de Moivre finds

$$P_a/(1-P_a)=\{p^b(p^a-q^a)\}/\{q^a(p^b-q^b)\}.$$

For further discussions of the history of this and the following problem the reader is referred to Thatcher (1957), Kohli (1975b) and Edwards (1983).

The problem may be considered as a random walk with two absorbing barriers. Seneta (1983) has pointed out that de Moivre's proof implies the notion of a martingale; see also Feller (1966, p. 214).

The duration of play; *Problems* 20–26. The problem of the duration of play is a continuation of the ruin problem. Under the same assumptions the question is: What is the probability that the play ends at the nth game or before?

De Moivre first discusses some special cases before he in Problem 24 gives his algorithm for finding the continuation probability. Let $u_n(x)$, $-a<x<b$, be the probability that A has $a+x$ counters after n games without A or B having been ruined during these games. The probability of continuation after n games is $U_n=\sum u_n(x)$ for $-a+1\leqslant x\leqslant b-1$. Let $a<b$, say. For $n<a$, $u_n(x)$ equals the corresponding term of the binomial expansion $(p+q)^n$. For $n=a$ we have to reject the extreme term q^a which corresponds to the ruin of A. For $n>a$ multiply the remaining sum by $p+q$ and continue until $(p+q)^n$ is reached, rejecting the extreme terms after each multiplication if the difference between the exponents of p and q equals either $-a$ or b. (This is obviously a modification of the algorithm leading to Pascal's triangle, but de Moivre does not mention this.)

In Problem 25 de Moivre assumes that A has infinitely many counters (a random walk with only one absorbing barrier) and without proof he gives the probability of B being ruined in at most n games as

$$\sum_{i=0}^{k}\binom{n}{i}p^{n-i}q^i+\sum_{i=k+i}^{n-b}\binom{n}{i+b}p^{n-i}q^i,\quad k=[(n-b)/2].$$

His formulation is somewhat obscure; his $n-d$ equals our b.

The distribution of the sum of a number of discrete, uniformly distributed random variables; *see the Lemma on p.* 220. The old problem of finding the distribution of the total number of points thrown with n dice is generalized by considering dice with f faces. De Moivre states the distribution without proof. His proof using the generating function $(t+t^2+\ldots+t^f)^n$ was first published in *Miscellanea Analytica* (1730, p. 196).

For the reader who wants to study how de Moivre treated the 26 problems in the three editions of *The Doctrine of Chances* we have prepared Table 1.

A referee has kindly informed me that a reprint of *De Mensura Sortis* was published by

Table 1

Correspondence between the 26 *problems in* '*De Mensura Sortis*' (1712) *and the problems in the three editions of* '*The Doctrine of Chances*' (1718, 1738, 1756).

1712	1718	1738	1756	Characterization of problem
1	1	p. 25	p. 25	The binomial distribution
2	2	p. 18	p. 18	The division problem for two players
3	3	1	1	
4	4	2	2	
5	5	3	3	Cumulative binomial equal to $\frac{1}{2}$. Find n. Approximative solution by means of Poisson distribution
6	6	4	4	
7	7	5	5	
8	8	6	6	The division problem for any number of players
9	9	7	7	The ruin problem. Huygens' fifth problem
10	10	8	8	The division problem for two players
11	11	10	10	Huygens' second problem
12	14	11	11	
13	—	—	—	Huygens' first problem
14	20	19	20	The hypergeometric distribution. Huygens' fourth problem
15	31	43	44	Waldegrave's problem
16	27	36	37	Robartes' problem. The division problem for two gamesters playing bowls
17	27	36	37	
18	29	38	39	The occupancy problem
19	30	39	40	
20	33	57	58	The duration of play. Algorithm for finding the probability that the duration is at least a given number of games, first for the players having the same numbers of counters and then for different numbers of counters. Also the case with one player having an infinite number of counters
21	35	59	60	
22	36	60	61	
23	37	61	62	
24	38	62	63	
25	40	64	65	
26	41	65	66	
p. 220	p. 17	p. 35	p. 39	Distribution of sum of discrete, uniformly distributed random variables

The numbers in the table are the problem numbers.

Kraus, New York, 1963, and that a translation into Italian by Dupont (1982) has recently been published.

3 Montmort's review of *De Mensura Sortis* and the priority dispute

Montmort received a copy of *De Mensura Sortis* from de Moivre in the beginning of August 1712. In a letter to N. Bernoulli of the 5th September 1712 he gave a detailed 'review' of de Moivre's work with comments on each of the 26 problems. This letter was published in Montmort's *Essay* (1713, pp. 361–370).

In 1712 de Moivre was 45 years old and a renowned mathematician. Obviously the 34-year old Montmort opened de Moivre's treatise with great expectations. However, Montmort expresses his disappointment and writes to N. Bernoulli that there is nearly nothing new for us in this treatise since the problems that are not already solved in my book have been solved in our letters. Even if this statement is true it is not fair to de Moivre since he did not and could not know these letters and therefore in good faith published his solutions as new. On the other hand Montmort adds that since de Moivre has based his work on Montmort's book there is nothing peculiar in the fact that further work leads both of them to similar results. Finally, referring to de Moivre's preface, Montmort writes that he would have wished—and it seems to him that equity demands so—that de Moivre frankly had recognized Montmort's priority and given him due credit. We shall make a few comments on the most important problems.

Montmort recognizes de Moivre's priority to the Poisson approximation, the division problem for two gamesters playing bowls (Robartes' problem) and the algorithm for the continuation probability in the problem on the duration of play. Teasingly he remarks about Robartes' problem that de Moivre has given the solution only for a special case and thus seems to have forgotten his criticism of Montmort on that score. Montmort then gives the solution for the case where the players are of the same skill but have a different number of bowls, see p. 367, and on pp. 248–251 he gives the general solution.

For all the other problems he points out, giving page references to the two editions of his book, that he himself and John and Nicholas Bernoulli have given solutions similar or even better than de Moivre's. Also in the preface does Montmort reply sharply against de Moivre's characterization of his work.

One would have expected an immediate reply from de Moivre but surprisingly this did not happen. Todhunter (1865, p. 187) writes:

> The publication of Montmort's second edition however does not seem to have produced any quarrel between him and De Moivre; the latter returned his thanks for the present of a copy of the work, and after this a frequent interchange of letters took place between the two mathematicians.

Furthermore, in the preface of *The Doctrine of Chances* (1718) de Moivre wrote:

> Since the printing of my Specimen (De Mensura Sortis), Mr. de Montmort, Author of the *Analyse des jeux de Hazard*, published a Second Edition of that Book, in which he has particularly given many proofs of his singular Genius, and extraordinary Capacity; which Testimony I give both to Truth, and to the Friendship with which he is pleased to Honour me.

In 1719 Montmort died at the age of 40. For de Moivre, however, the priority dispute was not over. In his *Miscellanea Analytica* (1730) he devoted a whole section to defend himself and the quotation about Montmort given above was not included in the second and third editions of *The Doctrine of Chances*.

Further comments on the relations between Montmort and de Moivre have been given by David (1962).

4 Notes on the translation

The title of *De Mensura Sortis* takes up p. 213 and the preface p. 214 of the *Phil. Trans.*, Vol. **27**. The paper itself is on pp. 215–264. Page references have been given in the translation in square brackets.

Printing errors in the text and obvious errors in the formulae have been corrected without notice apart from a few important cases where the correction has been stated.

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Résumé

Le traité *De Mensura Sortis* par A. de Moivre est traduit en anglais. Dans une commentaire on présente le fondement historique, un résumé en notation et terminologie moderne et quelques remarques de la dispute entre Montmort et de Moivre.

On the Measurement of Chance, or, on the Probability of Events in Games Depending Upon Fortuitous Chance

Abraham de Moivre, F.R.S.

Translated into English by Bruce McClintock

Subiaco, Western Australia, 6008, Australia

To that most distinguished gentleman Francis Robartes, most noble patron of the Mathematical Sciences.

By thy exhortation, most distinguished sir, I have solved certain problems pertaining to the die, and have revealed the principles by which their solution may be undertaken; and now I have published them by command of the Royal Society. *Huygens* was the first that I know who presented rules for the solution of this sort of problems, which a French author has very recently well illustrated with various examples; but these distinguished gentlemen do not seem to have employed that simplicity and generality which the nature of the matter demands: moreover, while they take up many unknown quantities, to represent the various conditions of gamesters, they make their calculation too complex; and while they suppose that the skill of gamesters is always equal, they confine this doctrine of games within limits too narrow. The method which I use especially is the Doctrine of Combinations, which being understood correctly, facilitates the solution of many otherwise very difficult problems; indeed I did not so restrict myself to this method that I did not several times employ infinite series, especially when priority of playing came to be considered. But these series are either broken off of themselves, or are summed up exactly, or tend to the truth. The three problems [Prob. 16, 17, 18] which thou, most renowned sir, didst propose that I should solve, I have completed, not without great pleasure; and if any praise should come to me from these things, I believe that it will be owed chiefly to their solution. If thou couldst, in the time which thou employest so profitably to the advantage of the State, follow those things which have been taken up to divert thy mind, and have been ascertained with very happy success, nothing should be wanting to the perfection of this doctrine; and at the same time it should be clear how thou excellest in a singular acuteness of intellect, and how contemplations of this sort may be not at all inconsistent with more severe things and with studies of greater moment.

Most distinguished sir,
Most observant of
and most obedient to thee,

Abr. De Moivre.

[p. 215] If p is the number of chances by which a certain event may happen, and q is the number of chances by which it may fail, the happenings as much as the failings have their degree of probability; but if all the chances by which the event may happen or fail were equally easy, the probability of happening will be to the probability of failing as p to q.

If A and B, two gamesters, were so to contend about events, that if p chances should happen A wins but if q chances should happen B wins, and there were a sum, a, placed down, the 'sors' or expectation of A will be $pa/(p+q)$ and the 'sors' or expectation of B

will be $qa/(p+q)$, and moreover, if A or B should sell his expectations, it is fair that they should receive $pa/(p+q)$ and $qa/(p+q)$ for them, respectively.

If it be proposed that a certain prize a should be granted to the winner, so that if p chances happen the prize should be granted to A, but if q chances happen the prize should be granted to B; and A and B enter into an agreement, that before the event the prize should be divided according to the odds, A ought to receive the portion $pa/(p+q)$, B the portion $qa/(p+q)$.

If two events have no dependence on each other, so that p is the number of chances by which the first event may happen and q the number of chances by which it may fail, and r is the number of chances by which the second event may happen and s the number of chances by which it may fail: multiply $p+q$ by $r+s$, and the product $pr+qr+ps+qs$ will contain all the chances by which the happenings of failings of the events may be varied amongst one another.

Therefore if A and B contend, and A should claim that both events would happen, the odds will be as pr to $qr+ps+qs$.

[p. 216] But if A should claim that one would happen, the odds will be as $pr+qr+ps$ to qs.

If, however, A should claim that the first event would happen and the second would fail, the odds will be as ps to $pr+qr+qs$.

And by the same method of arguing, if there should be three or more events about which A and B contend, the odds will be found by multiplication only.

If all events have a given number of chances by which they may happen, and likewise a given number of chances by which they may fail, and a is the number of chances by which each event may happen and b the number of chances by which it may fail, and n is the number of all events, let $a+b$ be raised to the nth power.

And if A plays with B on this condition, that if one or more events shall have happened A wins, but if none then B wins; the odds will be as $(a+b)^n-b^n$ to b^n, for the only term in which a is not found is b^n.

If A plays with B on this condition, that if two or more events shall have happened A wins, but if none or one then B wins, the odds will be as $(a+b)^n-b^n-nab^{n-1}$ to b^n+nab^{n-1}, for the two terms in which a^2 is not found are b^n and nab^{n-1}, and thus successively with the rest.

PROBLEM 1. A and B play with one die on this condition, that if in 8 throws of the die A throws an ace twice or more A wins, but if once only or not at all B wins; what are the odds?

Solution. Since there is but one chance by which an ace may happen and five chances by which it may fail, let $a=1$ and $b=5$. [p. 217] Again, since there are eight throws of the die, let $n=8$, and the odds will be $(a+b)^n-b^n-nab^{n-1}$ to b^n+nab^{n-1}, that is, as 663991 to 1015625 or roughly as 2 to 3.

PROBLEM 2. A and B play with single bowls on this condition, that he who shall have sent the bowl closest to the goal wins one game; now after a certain number of games are completed A lacks four games to win and B 6; but such is the skill of A in sending the bowls, that his chances are to the chances of B as 3 to 2 if they contest one game; what are the odds in the proposed case?

Solution. Since A lacks 4 games to win and B 6, it follows that the contest will be ended in at most 9 games, namely the sum of the games lacking minus one; therefore let $a+b$ be raised to the ninth power, this will be

$$a^9+9a^8b+36a^7b^2+84a^6b^3+126a^5b^4+126a^4b^5+84a^3b^6+36a^2b^7+9ab^8+b^9.$$

And all the terms in which a has 4 or more dimensions should be taken for A, and all the terms in which b has 6 or more dimensions for B, therefore the odds will be as

$$a^9+9a^8b+36a^7b^2+84a^6b^3+126a^5b^4+126a^4b^5 \text{ to } 84a^3b^6+36a^2b^7+9ab^8+b^9.$$

Let a be expressed by 3 and b by 2, and the odds will be had in numbers, namely 1759077 to 194048.

And generally, supposing that p and q are the numbers of games wanting respectively; raise $a+b$ to the power $p+q-1$, and let as many terms as games lacking to the adversary, respectively, be taken for A and B, that is, let as many terms as are units in q be taken for A, and as many terms as are units in p for B.

[p. 218] PROBLEM 3. If A and B play with single bowls, and the skill of A in sending bowls is such that he can give B two games out of three; it is sought what the odds are in any one game.

Solution. Let the odds that are sought be as z to 1 and raise $z+1$ to its cube, this is z^3+3z^2+3z+1. Now since A can give B two games out of three, A will be able to undertake to win three games successively, and in this case the odds will be as z^3 to $3z^2+3z+1$. Therefore $z^3=3z^2+3z+1$. Or $2z^3=z^3+3z^2+3z+1$. Therefore $z2^{1/3}=z+1$, and further $z=1/(2^{1/3}-1)$. Therefore the odds sought will be as $1/(2^{1/3}-1)$ to 1 respectively.

And generally, if the skill of A is such, that he may undertake with equal chance to win n turns successively, A will be able to lay $1/(2^{1/n}-1)$ to 1 that he will win in the first turn.

PROBLEM 4. If A can without advantage or disadvantage give B one out of three games, the odds of A and B are sought, when they contest one game; or what is the ratio of their skills.

Solution. Let the ratio of skills be as z to 1. If A can give B one game out of three, then A undertakes to win thrice, before B shall have won twice; let $z+1$ be raised to the [p. 219] fourth power, namely $z^4+4z^3+6z^2+4z+1$, therefore the odds will be as z^4+4z^3 to $6z^2+4z+1$; therefore when they contest with equal chance, let $z^4+4z^3=6z^2+4z+1$; this equation being solved, z will be found to be 1·6 very near. Therefore the ratio of skills will be nearly as 8 to 5.

PROBLEM 5. To find in how many trials it will be probable that an event will happen, supposing that there are a chances by which it may happen in the first trial, and b chances by which it may fail, so that if A and B contend about the event, A and B can affirm and deny the event with equal probability.

Solution. Let x be the number of trials by which an event may happen or fail with equal probability, therefore by what has been demonstrated $(a+b)^x-b^x=b^x$, or $(a+b)^x=2b^x$, therefore

$$x=\frac{\log 2}{\log(a+b)-\log b}.$$

Moreover, let us again take up the equation $(a+b)^x=2b^x$, and set $a:b::1:q$, and the equation changes to $(1+1/q)^x=2$. Let $1+1/q$ be raised to the xth power by Newton's Theorem, so that

$$1+\frac{x}{q}+\frac{x}{1}\frac{x-1}{2q^2}+\frac{x}{1}\frac{x-1}{2}\frac{x-2}{3q^3}+\ldots=2.$$

In this equation if $q=1$ then $x=1$; if q is infinite x will be infinite. Let x be infinite;

therefore the above equation will become

$$1+\frac{x}{q}+\frac{x^2}{2q^2}+\frac{x^3}{6q^3}+\ldots=2.$$

Again let $x/q=z$, then

$$1+z+\tfrac{1}{2}z^2+\tfrac{1}{6}z^3+\ldots=2.$$

But

$$1+z+\tfrac{1}{2}z^2+\tfrac{1}{6}z^3+\ldots$$

is the number of which the hyperbolic logarithm is z, therefore $z=\log 2$. But the hyperbolic logarithm of 2 is approximately 0·7, therefore $z=0{\cdot}7$ approximately.

[p. 220] Therefore when q is 1 then $x=1q$; and when q is infinite then $x=0{\cdot}7q$ approximately.

We have thus found the very narrow limits within which the ratio of x to q lies; for that ratio begins from unity, and when q has advanced to infinity it ends at last in the ratio of 7 to 10, approximately.

EXAMPLE 1. It must be found in how many throws A may undertake to throw two aces with two dice.

Solution. Since A has only one chance by which he may throw two aces and 35 by which he may miss them $q=35$; therefore let 35 by multipled by 0·7 and the product, 24·5, will indicate the number of throws sought to be between 24 and 25.

EXAMPLE 2. It must be found in how many throws A may undertake to throw three aces with three dice.

Solution. Since A has but one chance by which he may throw three aces with three dice and 215 chances by which he may miss them let 215 be multiplied by 0·7 and the product, 150·5, will indicate the number of throws sought to be between 150 and 151.

LEMMA. To find the number of chances by which a given number of points may be thrown with a given number of dice.

Solution. Let $p+1$ be the given number of points, n the number of dice, f the number of faces on each die, let $p-f=q$, $q-f=r$, [p. 221] $r-f=s$, $s-f=t$, etc. The number of chances will be

$$\begin{aligned}
&+\frac{p}{1}\frac{p-1}{2}\frac{p-2}{3}\ldots\\
&-\frac{q}{1}\frac{q-1}{2}\frac{q-2}{3}\ldots\frac{n}{1}\\
&+\frac{r}{1}\frac{r-1}{2}\frac{r-2}{3}\ldots\frac{n}{1}\frac{n-1}{2}\\
&-\frac{s}{1}\frac{s-1}{2}\frac{s-2}{3}\ldots\frac{n}{1}\frac{n-1}{2}\frac{n-2}{3}
\end{aligned}$$

etc. This series ought to be continued until any of the factors are either equal to zero or negative.

Note that as many factors in each single product,

$$\frac{p}{1}\frac{p-1}{2}\frac{p-2}{3}\cdots,\frac{r}{1}\frac{r-1}{2}\frac{r-2}{3}\cdots,\frac{s}{1}\frac{s-1}{2}\frac{s-2}{3}\cdots,$$

must be taken up as there are units in $n-1$.

Praxis. For instance, let the number of chances be required by which 16 points may be thrown with 4 dice:

$$+\frac{15}{1}\frac{14}{2}\frac{13}{3}=+455$$

$$-\frac{9}{1}\frac{8}{2}\frac{7}{3}\frac{4}{1}=\quad -336$$

$$+\frac{3}{1}\frac{2}{2}\frac{1}{3}\frac{4}{1}\frac{3}{2}=\quad +6.$$

Now $455-336+6=125$. Therefore the number of chances sought is 125.

Let the number of chances be required by which 15 points may be thrown with 6 dice:

$$+\frac{14}{1}\frac{13}{2}\frac{12}{3}\frac{11}{4}\frac{10}{5}=+2002$$

$$-\frac{8}{1}\frac{7}{2}\frac{6}{3}\frac{5}{4}\frac{4}{5}\frac{6}{1}=\quad -336.$$

Now $2002-336=1666$, the number of chances sought.

[p. 222] Let the number of chances be required by which 27 points may be thrown with 6 dice:

$$+\frac{26}{1}\frac{25}{2}\frac{24}{3}\frac{23}{4}\frac{22}{5}=\quad +65780$$

$$-\frac{20}{1}\frac{19}{2}\frac{18}{3}\frac{17}{4}\frac{16}{5}\frac{6}{1}=\quad -93024$$

$$+\frac{14}{1}\frac{13}{2}\frac{12}{3}\frac{11}{4}\frac{10}{5}\frac{6}{1}\frac{5}{2}=+30030$$

$$-\frac{8}{1}\frac{7}{2}\frac{6}{3}\frac{5}{4}\frac{4}{5}\frac{6}{1}\frac{5}{2}\frac{4}{3}=\quad -1120.$$

Now $65780-93024+30030-1120=1666$, the number of chances sought.

Corollary. All points equally distant from the extremes have the same number of chances by which they may be produced, and therefore, if the given number of points is closer to the greater extreme than to the less, let that number be subtracted from the sum of the extremes, and the number of chances may be found, by which the remaining number may be produced, and the operation may be made shorter.

Example 3. To find in how many throws A may undertake to throw 15 points with 6 dice.

Solution. Since A has 1666 chances by which he may throw 15 and 44990 by which he may miss them, let 44990 be divided by 1666 and the quotient, 27, will equal q. Therefore let 27 be multiplied by 0·7, and the product, 18·9, will indicate the number of throws sought to be nearly 19.

[p. 223] PROBLEM 6. To find in how many trials it will be probable that an event will happen twice, supposing that there are a chances by which it may happen in the first trial and b chances by which it may fail; so that if A and B contend about the event, A and B may affirm and deny the event with equal probability.

Solution. Let x be the number of trials, therefore by what has already been demonstrated, it will be manifest that $(a+b)^x = 2b^x + 2axb^{x-1}$. Or by making $a:b::1:q$,

$$\left(1+\frac{1}{q}\right)^x = 2+\frac{2x}{q}.$$

First, let $q=1$ then $x=3$. Secondly, let q be infinite and x will be infinite. Assume that x is infinite, and $x/q=z$, then

$$1+z+\tfrac{1}{2}z^2+\tfrac{1}{6}z^3+\ldots = 2+2z,$$

and therefore $z = \log 2 + \log(1+z)$; now if $\log 2$ is called y, that equation is changed into the fluxional $(z\dot{z})/(1+z) = \dot{y}$. And if the value of z is sought through the powers of y, it will be found that $z = 1{\cdot}678$ approximately, therefore x always lies between the limits $3q$ and $1{\cdot}678q$; but x very soon converges to $1{\cdot}678q$, and therefore if q does not have a very small ratio to 1 it may be assumed that $x = 1{\cdot}678q$. If there should be any suspicion that x is smaller than is correct, its value should be substituted in the equation

$$\left(1+\frac{1}{q}\right)^x = 2+\frac{2x}{q},$$

and the error should be noted; if there is any worthy of notice, then let x be increased a small amount, and the value so increased should be substituted for x in the aforesaid equation, and the new error should be noted, and by the aid of the two errors, the value of x may be corrected accurately enough.

[p. 224] EXAMPLE 1. It must be found in how many throws A may undertake twice to throw three aces with three dice.

Solution. Since A has but one chance by which he may throw three aces and 215 by which he may miss them $q=215$. Therefore let 215 be multipled by $1{\cdot}678$ and the product, $360{\cdot}7$, will indicate the number of throws sought to be between 360 and 361.

EXAMPLE 2. It must be found in how many throws A may undertake twice to throw 15 points with six dice.

Solution. Since A has 1666 chances by which he may throw 15 points and 44990 by which he may miss them let 44990 be divided by 1666, then the quotient, 27, $=q$. Therefore let 27 be multiplied by $1{\cdot}678$ and the product, $45{\cdot}3$, will indicate the number of throws sought to be between 45 and 46.

PROBLEM 7. To find in how many trials it will be probable that an event may happen thrice, four times, five times, etc., supposing that there are a chances by which it may happen in the first trial and b chances by which it may fail.

Solution. Let x be the number of trials sought, and from what has already been demonstrated if it is contended about a triple event, making $a:b::1:q$, then [p. 225]

$$\left(1+\frac{1}{q}\right)^x = 2\left(1+\frac{x}{q}+\frac{x}{1}\frac{x-1}{2q^2}\right).$$

If about a quadruple,

$$\left(1+\frac{1}{q}\right)^x=2\left(1+\frac{x}{q}+\frac{x}{1}\frac{x-1}{2q^2}+\frac{x}{1}\frac{x-1}{2}\frac{x-2}{3q^3}\right).$$

And the continuation of these equations is clear. Now in the first equation, if $q=1$ then $x=5q$, but if q is infinite, or has a large enough ratio to one, the aforesaid equation, assuming that $x/q=z$, shall change to $z=\log 2+\log(1+z+\frac{1}{2}z^2)$, or to this fluxional, supposing that $\log 2=y$:

$$\frac{\frac{1}{2}z^2\dot{z}}{1+z+\frac{1}{2}z^2}=\dot{y},$$

where it will be found that $z=2{\cdot}675$ approximately; therefore x always lies between $5q$ and $2{\cdot}675q$.

In the latter equation, if $q=1$ then $x=7q$, but if q is infinite, or has a large enough ratio to one, then

$$z=\log 2+\log(1+z+\tfrac{1}{2}z^2+\tfrac{1}{6}z^3) \quad \text{or} \quad \frac{\frac{1}{6}z^3\dot{z}}{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3}=\dot{y},$$

where it will be found that $z=3{\cdot}6719$ approximately; and similarly for the following, and the limits gradually approximate to the ratio of two to one.

Table of limits. If it is contended about a single event, the number of trials will be between

	$1q$ and $0{\cdot}693q$
If about a double, between	$3q$ and $1{\cdot}678q$
If about a triple, between	$5q$ and $2{\cdot}675q$
If about a quadruple, between	$7q$ and $3{\cdot}6719q$
If about a quintuple, between	$9q$ and $4{\cdot}67q$
If about a sextuple, between	$11q$ and $5{\cdot}668q$.

If it is contended about more events of which the number is n, provided n and q have a large enough ratio to one, a conjecture about the number of trials may be made easily, not straying much from the truth, by assuming that the number of trials $=\frac{1}{2}(2n-1)q$. Furthermore x soon converges to the smaller limit.

[p. 226] PROBLEM 8. Three gamesters, A, B, C, play with single bowls on this condition, that he who first wins a given number of games gains the stake; now after a certain number of games are completed A lacks 1 game, B 2, C 3; the ratio of skills are as a, b, c, respectively; the ratio of expectations is sought.

Solution. Let $a+b+c$ be raised to the fourth power (for indeed of necessity the contest will be ended in 4 games at most); this will be

$$a^4+4a^3b+6a^2b^2+4ab^3+b^4+4a^3c+12a^2bc+6a^2c^2+12abc^2+6b^2c^2+4ac^3+c^4$$
$$+4b^3c+12ab^2c+4bc^3.$$

The terms $a^4+4a^3b+12a^2bc+6a^2c^2+4a^3c+12abc^2$, where a rises to an equally high dimension as the number of games that A lacks or to a higher, and where b and c rise to lesser dimensions than the number of games that B and C lack, are part of A's expectation. By the same method the terms $b^4+4b^3c+6b^2c^2$ are part of B's expectation. And the terms $4bc^3+c^4$ are part of C's expectation. All the remaining terms are common and ought to be so divided that all the parts which favour one of the gamesters should be distributed to him.

Now since A lacks 1 game, B 2, C 3, all those terms in which a shall have reached the first or higher dimension before b shall have reached the second and c the third favour A, and in the same manner the parts which favour B and C are found, and so if the term $6a^2b^2$ be divided into its parts *aabb, abab, abba, baab, baba, bbaa,* the first 5 parts should be distributed to A, only the last part to B; therefore now $5a^2b^2$ ought to be added to A's expectation and $1a^2b^2$ to B's expectation. If the term $4ab^3$ be divided into its parts, *abbb, babb, bbab, bbba,* the first and second parts favour A, the third and fourth favour B, and so $2ab^3$ should be distributed to each. If the term $12ab^2c$ [p. 227] be divided into its parts 8 should be distributed to A and 4 parts to B; if the term $4ac^3$ be divided into its parts 3 parts should be distributed to A and but one part to C, and so the total expectations will now be:

first, $a^4+4a^3b+5a^2b^2+2ab^3+12a^2bc+4a^3c+6a^2c^2+8ab^2c+12abc^2+3ac^3$;

second, $b^4+4b^3c+6b^2c^2+a^2b^2+2ab^3+4ab^2c$;

third, $4bc^3+ac^3+c^4$.

[De Moivre's solution has been corrected in three places.]

Let n be the number of games lacking, p the number of gamesters, the ratio of chances as a, b, c, d, etc.; let $a+b+c+d$, etc. be raised to the power $n+1-p$ and proceed in the same way.

PROBLEM 9. A and B, taking 12 counters each, play with three dice on this condition, that if 11 points are thrown A should hand one counter to B, but if 14 points are thrown B should hand one counter to A, and that he wins the game who first holds all the counters: the ratio of A's probability to B's probability is sought.

Solution. Let p be the number of counters which each of them have; let a and b be the number of chances by which A and B respectively may obtain one counter, and the odds will be as a^p to b^p; in this case $p=12$, $a=27$, $b=15$; or since $27:15::9:5$, let $a=9$, $b=5$, and then the odds will be as 9^{12} to 5^{12}, or as 244140625 to 282429536481 as *Huygens* claimed it would be.

A more general solution. Let p be the number of A's counters and q the number of B's counters, and let A undertake to gain q counters [p. 228] before B gains p counters; the odds will be as $a^q(a^p-b^p)$ to $b^p(a^q-b^q)$. Suppose that A has the counters E, F, G, H, etc. of which the number is p, and B has the counters I, K, L, etc. of which the number is q, moreover suppose that the value of any counter is to the value of the following as a to b so that E, F, G, H, I, K, L are in geometric progression; supposing these things so, A and B will be able in any turn to lay down counters the value of which is proportional to the number of chances by which either may win, and so in the first turn A may lay down H and B I, but H is to I, from the hypothesis, as a to b, therefore A and B now play on equal terms; if A wins he will be able to lay down I and B K, but I is to K, from the hypothesis, as a to b; but if B wins A will be able to lay down G and B H, of which (G and H) the proportion is as a to b, and thus successively. Therefore as long as A and B play, they always play on equal terms. Therefore their expectations are as the sum of the terms E, F, G, H, etc. of which the number is p, to the sum of the terms I, K, L, of which the number is q; that is, as $a^q(a^p-b^p)$ to $b^p(a^q-b^q)$, which will be easily agreed to if those geometric progressions are summed up. The supposition that any counter is to the following as a to b does not change the probabilities of winning; therefore supposing that the counters have the same value, the probabilities of winning will be still in that same ratio which we have determined.

Beware greatly lest problems be confounded amongst themselves from any appearance of affinity. The following problem seems to have an affinity with the above.

[p. 229] Problem 10. Twenty-four counters are taken and C throws three dice; now as often as he shall have thrown 11 points he should hand over one counter to A, and as often as he shall have thrown 14 points he should hand over one counter to B, but A and B play with the agreement, that he who first shall get 12 counters should obtain the stake; the odds are sought.

This problem differs from the above in this, that in at most 23 throws of the dice, the game will be ended of necessity; while the game, by the rule of the preceding problem, may be continued indefinitely, because of the reciprocation of gains and losses which perpetually takes place.

Solution. Let $a+b$ be raised to the 23rd power and the first 12 terms will be to the last 12 as the expectation of A to the expectation of B.

Problem 11. Three gamesters A, B, C, taking 12 counters of which 4 are white and 8 black, play on this condition, that he who first of them shall have chosen, blindfold, a white counter wins, and that the first choice should be A's, the second B's, the third C's, and then following again A's and thus successively in order; it is sought what the ratio of expectations of A, B, C will be.

Solution. Let n be the number of all the counters, a the number of white ones, b the number of black, 1 the sum deposited or the prize to be granted to the winner.

[p. 230] (i) A has a chances by which he may choose a white and b chances by which he may choose a black, and so his expectation from the first choice must arise as $a/(a+b)$ or a/n. Therefore if a/n is subtracted from 1, the value of the remaining expectations will be $1-a/n=(n-a)/n=b/n$.

(ii) B has a chances by which he may choose a white, and $b-1$ chances by which he may choose a black, but the first choice is A's and it is uncertain whether he will be the winner or not and so the prize in respect of B is not 1 but only b/n, therefore his expectation from the second choice must arise as $(a/(a+b-1))(b/n)=(ab)/n(n-1)$. Let $ab/n(n-1)$ be subtracted from b/n, and the value of the remaining expectation will be $(nb-b-ab)/n(n-1)=b(b-1)/n(n-1)$.

(iii) C has a chances by which he may choose a white, and $b-2$ chances by which he may choose a black, and so his expectation from the third choice is $ab(b-1)/n(n-1)(n-2)$.

(iv) In the same way A has a chances by which he may choose a white and $b-3$ chances by which he may choose a black, and so his expectation from the fourth choice will be $ab(b-1)(b-2)/n(n-1)(n-2)(n-3)$. And thus successively for the rest.

Therefore the series may be written

$$\frac{a}{n}+\frac{b}{n-1}P+\frac{b-1}{n-2}Q+\frac{b-2}{n-3}R+\frac{b-3}{n-4}S+\dots,$$

where P, Q, R, S, etc. denote the preceding terms with their signs; and as many terms of this series should be assumed as there are units in $b+1$ (for there are no more choices than there are units in $b+1$). And the sum of every third term [p. 231] beginning from a/n will be A's whole expectation; likewise the sum of every third term beginning from $bP/(n-1)$ will be B's whole expectation; the sum of every third term beginning from $(b-1)Q/(n-2)$ will be C's whole expectation.

If there are more gamesters, A, B, C, D, etc. and they take one counter or more or the same number of counters or a different number, their expectations, by the aid of the preceding series, will likewise be determined easily.

But to revert to the case proposed in the problem let $a=4$, $b=8$, $n=12$ and the general series will now change into this

$$\frac{4}{12}+\frac{8}{11}P+\frac{7}{10}Q+\frac{6}{9}R+\frac{5}{8}S+\frac{4}{7}T+\frac{3}{6}V+\frac{2}{5}X+\frac{1}{4}Y.$$

Or into this other (by multiplying all the terms by that number which by removing the fractions will be judged more convenient, namely, in this case by 495)

$$165+120+84+56+35+20+10+4+1$$

and so to A will be attributed $165+56+10=231$; to B, $120+35+4=159$; to C, $84+20+1=105$. And so the expectations will be as 231, 159, 105; or as 77, 53, 35.

COROLLARY. If the number of cases by which A, B, C or any number of gamesters may win should be at any time at last exhausted, the expressions of expectations will be finite.

[p. 232] PROBLEM 12. If three gamesters, A, B, C, throw in their turns a dodecahedron with 4 white faces and 8 black on this condition, that he who first shall have thrown a white face wins, the ratio of expectations is sought.

Solution. The reasonings about this proposition are the same as those which we used on the preceding, but since the throws of a dodecahedron take nothing from the number of its faces for $b-1$, $b-2$, $b-3$, $b-4, \ldots$, $n-1$, $n-2$, $n-3$, $n-4, \ldots$, let b and n be substituted respectively, and the series of the preceding problem will become

$$\frac{a}{n}+\frac{ab}{n^2}+\frac{ab^2}{n^3}+\frac{ab^3}{n^4}+\frac{ab^4}{n^5}+\frac{ab^5}{n^6}+\ldots,$$

which series ought to be continued to infinity. And adding every third term the expectations will be

$$\frac{a}{n}+\frac{ab^3}{n^4}+\frac{ab^6}{n^7}+\ldots, \quad \frac{ab}{n^2}+\frac{ab^4}{n^5}+\frac{ab^7}{n^8}+\ldots, \quad \frac{ab^2}{n^3}+\frac{ab^5}{n^6}+\frac{ab^8}{n^9}+\ldots.$$

But the terms from which the individual expectations are composed are in geometric progression, and the ratio of any term to the following is the same in the individual series, namely as n^3 to b^3; therefore the sums of these series are as the first terms, that is as a/n, ab/n^2, ab^2/n^3, or as n^2, bn, b^2. That is, in the case of this problem, as 9, 6, 4.

COROLLARY. If there are more gamesters, A, B, C, D, etc. playing on the same conditions as above, as many terms in the ratio n to b should be taken as there are gamesters, and those terms denote the expectations of the gamesters respectively.

[p. 233] PROBLEM 13. A and B play with two dice on this condition, that A wins if he throws six, B if seven. First A makes one throw, then B two throws, then again A two throws, and thus successively, until the one or the other comes out the winner; the ratio of A's expectation to B's expectation is sought.

Solution. Let a be the number of chances by which A may win, and b the number of chances by which B may win, n the number of variations in the given dice, moreover let $n-a=d$ and $n-b=e$, and let 1 be the prize to be granted to the winner.

(i) A has a chances by which he may win and $n-a$ chances by which he may lose, and so his expectation from the first throw is a/n; therefore if a/n is subtracted from 1 the value of the remaining expectation will be $1-a/n=(n-a)/n=d/n$.

(ii) If B reaches his throw his expectation from this throw must be b/n, but since it is uncertain whether he will reach his throw or not the expectation b/n must be diminished in the ratio of d to n, and the prize in this respect must be considered not 1 but only d/n, and so B's expectation before A makes his throw will be bd/n^2; let bd/n^2 be subtracted from d/n and the value of the remaining expectation will be $d/n - bd/n^2 = (nd - bd)/n^2 = ed/n^2$.

(iii) By the same method of arguing B's expectation subsequent to this is bed/n^3.

[p. 234] (iv) And the expectation of A subsequent to this is ae^2d/n^4.

(v) And the expectation of A subsequent to this is ae^2d^2/n^5. And thus successively for the rest; and so all the expectations of A will be

$$\begin{aligned} &a/n \\ &+ae^2d/n^4 + ae^2d^2/n^5 \\ &+ae^4d^3/n^8 + ae^4d^4/n^9 \\ &+ae^6d^5/n^{12} + ae^6d^6/n^{13} \\ &\text{etc.} \end{aligned}$$

Now, if we separate for a little time the first term a/n, the first perpendicular column constitutes an infinitely decreasing geometric progression of which the sum is $ae^2d/(n^4 - e^2d^2)$. Let the first term a/n be brought back and let it be added to the sum of the progression and the aggregate will be $(nae^2d + an^4 - ae^2d^2)/n(n^4 - e^2d^2)$.

The second column constitutes another geometric progression of which the sum is $ae^2d^2/n(n^4 - e^2d^2)$.

Therefore the sum of A's expectations is $(ae^2d + an^3)/(n^4 - e^2d^2)$.

All the expectations of B are

$$\begin{aligned} &bd/n^2 + bed/n^3 \\ &+be^2d^3/n^6 + be^3d^3/n^7 \\ &+be^4d^5/n^{10} + be^5d^5/n^{11} \\ &\text{etc.} \end{aligned}$$

[p. 235] The sum of the first column is $bdn^2/(n^4 - e^2d^2)$.

The sum of the second column is $bden/(n^4 - e^2d^2)$.

And so the sum of B's expectations will be $(bdn^2 + bden)/(n^4 - e^2d^2)$.

Therefore the ratio of expectations will be as $ae^2d + an^3$ to $bdn^2 + bden$.

If for a, b, n, d, e, are written 5, 6, 36, 31, 30, respectively, the ratio sought will be expressed in numbers, namely as 10355 to 12276.

Corollary. If the number of cases by which the gamesters may win is never exhausted so that the game may be continued forever, however as the gamesters because of that continuation may be placed at some time in the same circumstances in which they had been before, the expressions of the expectations are finite.

Problem 14. Taking 12 counters, 4 white and 8 black, A contends with B that if he shall have taken out, blindfold, 7 counters 3 of them would be white; the ratio of the expectation of A to the expectation of B is sought.

Solution. (i) Let all the chances by which 7 counters may be taken out from 12 be found, there will be 792 chances, as is plain from the rule of combinations:

$$\frac{12}{1}\,\frac{11}{2}\,\frac{10}{3}\,\frac{9}{4}\,\frac{8}{5}\,\frac{7}{6}\,\frac{6}{7} = 792.$$

(ii) Let 3 whites be separated, and let all the chances by which 4 blacks out of 8 may be joined to them be found; there will be 70 chances:

$$\frac{8}{1}\frac{7}{2}\frac{6}{3}\frac{5}{4}=70.$$

[p. 236] And since there are 4 chances by which 3 whites may be chosen from 4 let 70 be multipled by 4 and so there will be 280 chances by which 3 whites may be taken out with 4 blacks.

(iii) By the rule of games, he who undertakes to produce any effect will still be considered the winner, if he shall have produced the effect more times than he shall have undertaken, unless the contrary should have been expressly ordered, and so if 4 whites are taken out with 3 blacks, A must be considered the winner; therefore let 4 whites be separated, and let all the chances by which 3 blacks out of 8 may be joined to 4 whites be found, there will be 56 chances:

$$\frac{8}{1}\frac{7}{2}\frac{6}{3}=56.$$

(iv) Therefore A has $280+56=336$ chances by which he may come out the winner; let these chances be subtracted from 792 and there will be 456 remaining chances by which B may come out the winner; therefore the ratio of A's expectation to B's expectation will be as 336 to 456, or as 14 to 19.

Generally. Let n be the number of all the counters, a the number of whites, b the number of blacks, c the number which A takes out; and the Number of All the Chances will be

$$\frac{n}{1}\frac{n-1}{2}\frac{n-2}{3}\frac{n-3}{4}\frac{n-4}{5}\frac{n-5}{6}\cdots,$$

which series ought to be continued for as many terms as there are units in c.

The number of chances by which A may take out c counters without any white is

$$\frac{b}{1}\frac{b-1}{2}\frac{b-2}{3}\frac{b-3}{4}\frac{b-4}{5}\frac{b-5}{6}\cdots.$$

The number of chances by which A may take out one white counter is

$$\frac{b}{1}\frac{b-1}{2}\frac{b-2}{3}\frac{b-3}{4}\frac{b-4}{5}\cdots\frac{a}{1}.$$

[p. 237] The number of chances by which A may take out two white counters is

$$\frac{b}{1}\frac{b-1}{2}\frac{b-2}{3}\frac{b-3}{4}\cdots\frac{a}{1}\frac{a-1}{2}.$$

The number of chances by which A may take out three white counters is

$$\frac{b}{1}\frac{b-1}{2}\frac{b-2}{3}\cdots\frac{a}{1}\frac{a-1}{2}\frac{a-2}{3}.$$

The number of chances by which A may take out four white counters is

$$\frac{b}{1}\frac{b-1}{2}\cdots\frac{a}{1}\frac{a-1}{2}\frac{a-2}{3}\frac{a-3}{4}.$$

And thus successively.

PROBLEM 15. A, B, C, three gamesters, whose skills are equal, each lay down 1 and play on these conditions: (i) that two of them begin the game; (ii) that the loser should give his place to a third, so that the third may now contend with the winner, which condition must always be observed afterwards; (iii) that the loser should always be fined a sum p which serves for increasing the stake; (iv) that he should obtain all the stake thus gradually increased, who shall have defeated the two others successively. It is sought by how much the expectation of A and B, whom we have supposed to begin the game, is greater or smaller than that of C.

Solution. Let it be supposed that the game will be continued forever in the following way:

[p. 238] A defeats B		$3+p$
C defeats A	the stake	$3+2p$
B defeats C		$3+3p$
A defeats B		$3+4p$
C defeats A	the stake	$3+5p$
B defeats C		$3+6p$
A defeats B		$3+7p$
C defeats A	the stake	$3+8p$
B defeats C		$3+9p$
etc.		etc.

Let R be a spectator who, after A has defeated B once, asks A whether he wishes to sell to him the sums which he hopes to obtain and at how much he values them, consenting to which A replies:

'Since I have just defeated B I have an equal chance as to whether or not I obtain $3+2p$ and so that sum is worth $(3+2p)/2$.

'If C shall have happened to defeat me, yet however my turn for playing with C may come back, then I shall have an equal chance as to whether or not I obtain $3+5p$ and so the expectation of defeating C at that time will be worth $(3+5p)/2$. But since there are 7 against 1 that that turn will not come back (for C ought to defeat me, B defeat C, I B again), that sum which I hope to obtain is worth $(3+5p)/2\times 8$'.

In the same way, A observes, undertaking the computation again, that the value of the succeeding sum which he hopes to obtain is $(3+8p)/2\times 8\times 8$. And of the following $(3+11p)/2\times 8\times 8\times 8$. And thus infinitely.

[p. 239] R, ascertaining that this computation is just, pays A the sums

$$\frac{3+2p}{2}, \frac{3+5p}{2\times 8}, \frac{3+8p}{2\times 8\times 8}, \frac{3+11p}{2\times 8\times 8\times 8}, \ldots,$$

which by the use of the following theorem may be reduced to one sum.

THEOREM. If we sum to infinity

$$\frac{n}{b}+\frac{n+d}{b^2}+\frac{n+2d}{b^3}+\frac{n+3d}{b^4}+\ldots=\frac{n}{b-1}+\frac{d}{(b-1)^2}.$$

Let the series

$$\frac{3+2p}{2}+\frac{3+5p}{2\times 8}+\ldots$$

be distinguished in two parts

$$\frac{3}{2}\left(1+\frac{1}{8}+\frac{1}{8\times 8}+\frac{1}{8\times 8\times 8}+\frac{1}{8\times 8\times 8\times 8}+\dots\right)$$
$$+\frac{p}{2}\left(2+\frac{5}{8}+\frac{8}{8\times 8}+\frac{11}{8\times 8\times 8}+\frac{14}{8\times 8\times 8\times 8}+\dots\right).$$

The first part constitutes a geometric progression of which the sum is $\frac{12}{7}$.

The second part, if we separate the common multiplier $p/2$ and the first term 2, may be summed through the theorem set out above and becomes $\frac{5}{7}+\frac{3}{49}=\frac{38}{49}$; by adding the first 2 to this, the sum will be $\frac{136}{49}$; multiplication of which by $p/2$, the product, $68p/49$, will show the sum of the second series. Therefore R pays A $\frac{12}{7}+\frac{68}{49}p$.

R, turning to B in the same way, asks him whether he wants to sell him the sums which he is hoping to obtain, assenting to which B, relying on the same reasoning as did A, asks for the sum $\frac{3}{7}+\frac{31}{49}p$, which R perceiving it to be just pays to B.

Finally R, entering into the same agreement with C, pays him for the sums which he hopes to obtain $\frac{6}{7}+\frac{48}{49}p$.

[p. 240] Let S be another spectator, whom A asks (after he has defeated B once) whether he wishes to sustain his losses, that is, whether he wishes to be fined the sums p on behalf of A as often as it happens that he must be fined and for how much money he would wish to undertake this lot, to which S responds:

> 'Since you have an equal chance as to whether you defeat C or not, and thus whether you may be fined the sum p or not, I shall sustain the lot of this fine if you give into my hand $p/4$.
>
> 'But if C happens to defeat you, and B defeats C, so that in a second turn you must play with C, then I shall likewise sustain the lot of that same fine if you give me $p/2$; but since there are 3 against 1 that that will not happen, I shall sustain the lot of that fine if you now give into my hand $p/8$.
>
> 'And by the same method of arguing for the next lot to this if you give me $p/16$.
>
> 'And successively the next to this if you give me $p/64$, etc.'.

Now A, agreeing with S, hands over to S the sums

$$\tfrac{1}{2}p+\tfrac{1}{8}p+\tfrac{1}{16}p+\tfrac{1}{64}p+\tfrac{1}{128}p+\tfrac{1}{512}p+\tfrac{1}{1024}p+\dots,$$

which sums reduced to one make $5p/7$.

And in the same way B and C enter into an agreement with S and hand over to him $3p/7$ and $6p/7$, respectively, to sustain their lots of fines.

A receives from R	$\frac{12}{7}+\frac{68}{49}p$
A hands over to S	$\frac{35}{49}p$
There remains to A	$\frac{12}{7}+\frac{33}{49}p$.

But A had deposited 1 before he began the game, therefore A gains $\frac{5}{7}+\frac{33}{49}p$.

[p. 241] B receives from R	$\frac{3}{7}+\frac{31}{49}p$
B hands over to S	$\frac{21}{49}p=\frac{3}{7}p$
There remains to B	$\frac{3}{7}+\frac{10}{49}p$.

But B had deposited $1+p$ (namely 1 before the game began, and p after he had been defeated once by A); therefore B gains $-\frac{4}{7}-\frac{39}{49}p$.

Therefore the sum of the gains of A and B is $\frac{1}{7}-\frac{6}{49}p$.

Now we had supposed that A defeated B once before the gamesters entered into the agreement with R and S; but before the game began, B could look forward with equal chance to defeat A, and so the sum of gains $\frac{1}{7}-\frac{6}{49}p$ ought to be divided into equal portions so that the gain of each ought to be considered $\frac{1}{14}-\frac{3}{49}p$.

Therefore it may now be concluded that C's loss is $\frac{1}{7}-\frac{6}{49}p$ or his gain $-\frac{1}{7}+\frac{6}{49}p$.

But so that our computation may be corroborated let us see what C's gain should be by the same method which we used for finding the gains of A and B:

C receives from R	$\frac{6}{7}+\frac{48}{49}p$
C hands over to S	$\frac{42}{49}p$
There remains to C	$\frac{6}{7}+\frac{6}{49}p$.

But C had deposited $\frac{7}{7}$. Therefore C gains $-\frac{1}{7}+\frac{6}{49}p$.

[p. 242] Now let $\frac{1}{7}-\frac{6}{49}p=0$, and it will be found that $p=\frac{7}{6}$; therefore if the fine is to the sum which they each deposited as 7 to 6 the gamesters play on equal terms.

If the fine is in a proportion less than 7 to 6 to the sum which each deposited A and B will play on better terms, C on worse.

If the fine is in a proportion greater than 7 to 6 to the sum which each deposited A and B play on worse terms, C on better.

Corollary 1. After A has defeated B once the probabilities of winning will be as $\frac{12}{7}$, $\frac{6}{7}$, $\frac{3}{7}$ or as 4, 2, 1, so that the greatest probability is A's, the next C's, the least B's.

Corollary 2. Before the game begins the spectator R will be able to undertake to pay the sum for which the 3 gamesters contend, and all the fines, if at the beginning there was given into his hand $3+3p$.

Corollary 3. If the skills of the gamesters are in a given proportion, the expectations of the gamesters will be determined by the same reasoning.

Corollary 4. If the fine is negative so that the loser takes a small portion of the deposit, 3, for instance $\frac{3}{10}$, and the game must be ended when the stake has been exhausted, the expectations of the gamesters will be determined by the same reasoning.

Corollary 5. If there are more gamesters, $A, B, C, D, \ldots$, and they do not leave the game before one of them shall have defeated all the others successively, the ratio of expectations may still be found.

[p. 243] Corollary 6. If the fine is not constant, but continually increases or decreases by a rule that shall have been agreed upon, the ratio of expectations may still be determined, if not by finite expressions yet at least by a series perpetually converging to the truth.

Problem 16. A and B, whose skills are equal, play with a given number of bowls; now after a certain number of games are completed, A lacks 1 game to win and B 2; the ratio of their expectations is sought.

Solution. Let m be the number of all the bowls so that each has $m/2$; let p be the number of chances by which two or more out of B's bowls may lie closer to the jack; let q be the number of chances by which one or more out of B's bowls may lie closer to the jack so that $q-p$ is the number of chances by which one out of B's bowls (excluding more) may lie closer to the jack; let s be the number of all the variations which all bowls may undergo; let 1 be the whole stake.

It is clear that B has p chances by which he may obtain 1 and $q-p$ chances by which he may obtain $\frac{1}{2}$ and thus that his expectation is

$$(p+\tfrac{1}{2}q-\tfrac{1}{2}p)/s=(\tfrac{1}{2}p+\tfrac{1}{2}q)/s.$$

Now it is agreed from the Doctrine of combinations that all the bowls m may be varied in $m(m-1)(m-2)(m-3)\ldots$ ways, which series ought to be continued until the last term should become equal to unity and so $s=m(m-1)(m-2)(m-3)\ldots.$

It is agreed from the same Doctrine that two out of $\frac{1}{2}m$ bowls may be permuted in $\frac{1}{2}m(\frac{1}{2}m-1)$ ways, while all the remaining bowls [p. 244] of A and B, of which the number is $m-2$, may be varied in $(m-2)(m-3)(m-4)\ldots$ ways and so $p=\frac{1}{2}m(\frac{1}{2}m-1)(m-2)(m-3)(m-4)\ldots.$ Therefore

$$s:p::m(m-1):\tfrac{1}{2}m(\tfrac{1}{2}m-1)::(m-1):\left(\frac{m}{4}-\frac{1}{2}\right),$$

and $p=(ms/4-s/2)/(m-1)$.

It is clear that $\frac{1}{2}m$ bowls may be taken up one by one in $\frac{1}{2}m$ ways, while all the remaining bowls of A and B, of which the number is $m-1$, may be varied in $(m-1)(m-2)(m-3)(m-4)\ldots$ ways, and thus that $s:q::m:\frac{1}{2}m::1:\frac{1}{2}$; therefore $q=\frac{1}{2}s=(\frac{1}{2}ms-\frac{1}{2}s)/(m-1)$.

Therefore

$$\frac{\frac{1}{2}p+\frac{1}{2}q}{s}=\frac{\frac{3}{8}m-\frac{1}{2}}{m-1};$$

let this be subtracted from 1 and the remainder $(5m/8-\frac{1}{2})/(m-1)$ will be the expectation of A, and so the ratio of expectations of A and B will be as $5m/8-\frac{1}{2}$ to $3m/8-\frac{1}{2}$ or as $5m-4$ to $3m-4$.

COROLLARY 1. If the number of bowls was infinite the ratio of expectations would become as 5 to 3.

COROLLARY 2. If the skills are in a given proportion the ratio of expectations will be found by the same reasoning.

[p. 245] PROBLEM 17. A and B, whose skills are equal, play with a given number of bowls; now after a certain number of games are completed A lacks one game to win and B 3; the ratio of expectations of A and B is required.

Solution. As in the preceding problem let m be the number of all the bowls; let r be the number of chances by which 3 or more out of B's bowls may lie closer to the jack, p the number of chances by which 2 or more, q the number of chances by which 1 or more may lie closer to the jack; let s be the number of all the variations which all the bowls may undergo.

Therefore B has r chances by which he may obtain 1, $p-r$ chances by which he may obtain $\frac{1}{2}$, and $q-p$ chances by which he may obtain $(3m-4)/(8m-8)$ as is clear from the preceding, and so the sum of his expectations will be

$$\{r\times 1+(p-r)\tfrac{1}{2}+(q-p)(3m-4)/(8m-8)\}/s=\{\tfrac{1}{2}r+\tfrac{1}{2}p+(q-p)(3m-4)/(8m-8)\}/s.$$

Now three out of $\frac{1}{2}m$ bowls may be permuted in $\frac{1}{2}m(\frac{1}{2}m-1)(\frac{1}{2}m-2)$ ways, while all the remaining bowls of A and B, of which the number is $m-3$, may be varied in $(m-3)(m-4)\ldots$ ways. Therefore $r=\frac{1}{2}m(\frac{1}{2}m-1)(\frac{1}{2}m-2)(m-3)(m-4)\ldots.$ But $s=m(m-1)(m-2)(m-3)(m-4)\ldots.$ Therefore $r=(ms/8-\frac{1}{2}s)/(m-1)$.

But from the preceding problem $p=(ms/4-s/2)/(m-1)$ and $q=(\frac{1}{2}ms-\frac{1}{2}s)/(m-1)$.

[p. 246] Therefore, by substituting these values for r, p, q, B's expectation becomes equal to $(9m^2-26m+16)/(32m^2-64m+32)$. Let this be subtracted from 1 and A's expectation will be equal to $(23m^2-38m+16)/(32m^2-64m+32)$; and so the ratio of the expectations of A and B will be as $23m^2-38m+16$ to $9m^2-26m+16$ for any number of bowls at all excepting two.

To find the ratio of expectations of A and B when they play with single bowls or when the number of bowls is 2, let the general expression of B's expectation, namely

$$\{\tfrac{1}{2}r+\tfrac{1}{2}p+(q-p)(3m-4)/(8m-8)\}/s,$$

again be taken up and set r and p equal to 0, and B's expectation will be equal to

$$q\frac{3m-4}{8m-8}\Big/s=\frac{\frac{1}{2}m-\frac{1}{2}}{m-1}\frac{3m-4}{8m-8}=\tfrac{1}{2}\times\tfrac{2}{8}=\tfrac{1}{8},$$

which being subtracted from 1 makes A's expectation equal to $\frac{7}{8}$; therefore the proportion of expectations of A and B in this case will be as 7 to 1, which is agreed from principles long since set out elsewhere.

Corollary 1. If the number of bowls was infinite the ratio of expectations would become as 23 to 9.

Corollary 2. If A lacks any number of games to win and B likewise lacks any number of games the ratio of expectations may be discovered by the same reasoning.

Corollary 3. If the skills are in a given proportion the ratio of expectations may also be found.

[p. 247] Problem 18. A contends with B that he shall throw, in a given number of trials with a die having a given number of faces, certain given faces; A's expectation is sought.

Solution. Let $p+1$ be the number of faces on the die, n the given number of trials, f the number of faces which he must throw.

The number of chances by which A may throw an ace once or more in n trials is $(p+1)^n-p^n$ as is clear from what has previously been demonstrated.

Let the deuce be expunged from the number of faces so that the number of faces is reduced to p; then the number of chances by which A may throw an ace once or more times in n trials will be $p^n-(p-1)^n$.

Therefore, with the deuce restored, the number of chances by which A may throw an ace and a deuce is the difference of these chances, namely $(p+1)^n-2p^n+(p-1)^n$.

Now let the three be expunged; then the number of chances by which A may throw an ace and a deuce will be $p^n-2(p-1)^n+(p-2)^n$.

Therefore, with the three restored, the number of chances by which A may throw an ace, a deuce and a three is $(p+1)^n-3p^n+3(p-1)^n-(p-2)^n$. And thus successively for the rest.

Therefore let all the powers be written in order (the signs changing alternately), $(p+1)^n-p^n+(p-1)^n-(p-2)^n+(p-3)^n$ etc. And let the coefficients of a binomial raised to the power f be prefixed to these, and the sum of the terms will be the numerator of A's expectation, the denominator of which will be $(p+1)^n$.

[p. 248] Example 1. Let 6 be the number of faces on the die and 2 the number of given faces which ought to be thrown in 8 trials, then A's expectation will be $(6^8-2\times5^8+4^8)/6^8$.

Example 2. Let 6 be the number of faces on the die and 6 the number of faces which ought to be thrown in 12 trials; then A's expectation will be

$$(6^{12}-6\times5^{12}+15\times4^{12}-20\times3^{12}+15\times2^{12}-6\times1^{12})/6^{12}.$$

EXAMPLE 3. A contends with B that he shall throw two given faces in 43 trials with a die having 36 faces, or that he shall throw two aces at the same time on two common dice and also two deuces at the same time, then A's expectation will be

$$(36^{43}-2\times 35^{43}+34^{43})/36^{43}.$$

Note that computation of the parts of which these expectations are composed by addition and subtraction will be easy with the help of a Table of Logarithms.

PROBLEM 19. To find in how many trials it will be probable that A, one of two gamesters, will throw certain given faces with a die having a given number of faces.

Solution. As before, let $p+1$ be the number of faces on the die, f the given number of faces, n the number of trials sought. [We have here corrected de Moivre's formulation since he has confounded n with f]. Set

$$\log\frac{1}{1-(\frac{1}{2})^{1/f}}=\alpha,\quad \log\{(p+1)/p\}=\beta$$

and it will be found that $n=\alpha/\beta$ nearly.

[p. 249] *Demonstration.* If the number of faces which must be thrown is 6, A's expectation will be

$$\{(p+1)^n-6p^n+15(p-1)^n-20(p-2)^n+15(p-3)^n-6(p-4)^n+(p-5)^n\}/(p+1)^n.$$

Suppose that the terms $p+1$, p, $p-1$, $p-2$, etc. are in geometric progression, which supposition will not stray much from the truth, especially if p has a large enough ratio to 1, and set $p^n/(p+1)^n=1/r^n$; then A's expectation will be

$$1-\frac{6}{r^n}+\frac{15}{r^{2n}}-\frac{20}{r^{3n}}+\frac{15}{r^{4n}}-\frac{6}{r^{5n}}+\frac{1}{r^{6n}}=\tfrac{1}{2}.$$

Let the sixth root be extracted on both sides and it will become $1-1/r^n=(\frac{1}{2})^{1/6}$, therefore $r^n=1/(1-(\frac{1}{2})^{1/6})$; now with

$$\log r=\beta,\quad \log\{1/(1-(\tfrac{1}{2})^{1/6})\}=\alpha,$$

it will be seen that $n\beta=\alpha$ and so $n=\alpha/\beta$, and the demonstration for the remaining cases will be the same.

If there should be any suspicion lest the value of the index n thus found is not accurate enough, then let this value be substituted for n and the error be noted, then let this value be changed a small amount and the new error be noted, and with the aid of the two errors the value of the index n will be accurately enough corrected if the Regula falsi is applied.

The value of the index n thus found may be corrected through an infinite series, produced from the nature of the problem, such that the first term of this series is that value which we have assigned; but correction by the difference of errors suffices for practical uses.

EXAMPLE 1. To find in how many throws of a common die it is probable that A may throw all the faces.

[p. 250]

$$\log\frac{1}{1-(\frac{1}{2})^{1/6}}=0{\cdot}9621753,\quad \log\tfrac{6}{5}=0{\cdot}0791812,$$

therefore

$$n=\frac{0{\cdot}9621753}{0{\cdot}0791812}=12+.$$

Therefore it is now possible to conclude that the number of throws sought will be around 12, and if 12 is substituted for n in the equation suitable to this case, A's expectation will be found to be nearly 0·437 which is somewhat less than it ought to be, namely 0·5; therefore let 13 be put for n, and A's expectation will be found to be 0·513 which is more than it ought to be; therefore A will be able to undertake to throw all the faces in 13 trials and that on a stronger basis.

EXAMPLE 2. To find in how many trials it will be probable that A may throw six given faces with a die having 216 faces, or that he may throw all three [Raffles] with three common dice.

$$\log\frac{1}{1-(\frac{1}{2})^{1/6}}=0{\cdot}9621753, \quad \log\frac{216}{215}=0{\cdot}0020152;$$

therefore

$$n=\frac{0{\cdot}9621753}{0{\cdot}0020152}=\text{nearly } 477.$$

[p. 251] **On the duration of play**

PROBLEM 20. A and B, whose skills are in a given proportion, namely as a to b, play on this condition: that as often as A wins a game, B shall hand over to him one counter; and as often as B wins, A shall hand over one counter to him; and that they should not end the play before one of them has gained all the other's counters. Moreover two spectators, R and S, stand by, of whom R claims that the contest will be completed within a given number of games, S denies; the expectation of S is sought.

Solution, Case 1. Let 2 be the number of counters which each of the gamesters has, and let 2 also be the number about which R and S contend; now because 2 is the number of games about which there is contention, let $a+b$ be raised to the power 2 which will give $a^2+2ab+b^2$; the term $2ab$ favours S, the rest are adverse and so his expectation will be $2ab/(a+b)^2$.

[p. 252] *Case* 2. Let 2 be the number of counters which each of the gamesters has, and let 3 be the number of games about which R and S contend; therefore, let $a+b$ be raised to the third power which will give $a^3+3a^2b+3ab^2+b^3$. Now the two terms $+a^3+b^3$ are both adverse to S, the two remaining, $3a^2b+3ab^2$, partly favour, partly are adverse; therefore let these terms be divided into their parts, namely $3a^2b$ into aab, aba, baa, and $3ab^2$ into abb, bab, bba, and the parts $aba+baa+abb+bab$ or $2a^2b+2ab^2$ favour S, the rest are adverse. And so the expectation of S will be $(2a^2b+2ab^2)/(a+b)^3$ or (with the numerator and denominator divided by $a+b$), $2ab/(a+b)^2$, which is the same as in the preceding case.

Case 3. Let 2 be the number of counters which each of the gamesters has, and 4 the number of games about which the spectators contend; therefore let $a+b$ be raised to the fourth power which gives $a^4+4a^3b+6a^2b^2+4ab^3+b^4$; the terms $a^4+4a^3b+4ab^3+b^4$ are all adverse to S, only the term $6a^2b^2$ partly favours, partly is adverse to him; therefore let that term be divided into its parts, $aabb$, $abab$, $abba$, $baab$, $baba$, $bbaa$, and four parts, $abab$, $abba$, $baab$, $baba$, or $4a^2b^2$, favour S; and so his expectation will be $4a^2b^2/(a+b)^4$.

Case 4. Let 2 be the number of counters which each of the gamesters has, and 5 the number of games about which the spectators contend, and the expectation of S will be found to be the same as in the preceding case.

[p. 253] *Case* 5. Let 2 be the number of counters which each of the gamesters has, and 6 the number of games about which the spectators contend, and the expectation of S will be found to be $8a^3b^3/(a+b)^6$.

Generally. Let 2 be the number of counters which each of the gamesters has, and $2+d$ the number of games about which the spectators contend, then the expectation of S will be $(2ab)^{1+\frac{1}{2}d}/(a+b)^{2+d}$.

Note that here d is an even number, but if d is an odd number the expectation of S will be the same as if that number had been diminished by one.

Case 6. Let 3 be the number of counters which each of the gamesters has, and $3+d$ the number of games about which the spectators contend, then the expectation of S will be $(3ab)^{1+\frac{1}{2}d}/(a+b)^{2+d}$.

Note that here d is an even number, but if d is an odd number the expectation of S will be the same as if that number had been diminished by one.

Case 7. Let 4 be the number of counters which each of the gamesters has, and 4 the number of games about which the spectators contend, then the expectation of S will be $(4a^3b+6a^2b^2+4ab^3)/(a+b)^4$.

[p. 254] *Case* 8. Let 4 be the number of counters which each of the gamesters has, and 6 the number of games about which the spectators contend, then the expectation of S will be $(14a^4b^2+20a^3b^3+14a^2b^4)/(a+b)^6$.

A table of S's expectations for 4 counters

4	$(4a^3b+6a^2b^2+4ab^3)/(a+b)^4$
6	$(14a^4b^2+20a^3b^3+14a^2b^4)/(a+b)^6$
8	$(48a^5b^3+68a^4b^4+48a^3b^5)/(a+b)^8$
10	$(164a^6b^4+232a^5b^5+164a^4b^6)/(a+b)^{10}$
12	$(560a^7b^5+792a^6b^6+560a^5b^7)/(a+b)^{12}$
	etc.

The table of S's expectations will be easily continued if the following things are noted:

(i) that the coefficient of the first term in any numerator is the sum of the coefficients of all the terms in the preceding numerator;

(ii) that the coefficient of the second term is the aggregate of that sum and of the coefficient of the second term of the preceding;

(iii) that the coefficient of the third term is the same as the coefficient of the first term;

(iv) that the literal products may be formed by multiplying the preceding by ab, the first from the first, the second from the second;

(v) that all the denominators are that power of the binomial $a+b$, which is designated by the number of games about which R and S contend.

[p. 255] Here incidentally it comes to be observed that all the coefficients can be generated, the first from the first, the second from the second. Further if the penultimate, being doubled, is subtracted from the preceding ultimate multiplied by four, the required coefficient will arise.

General Rule. Let n be the number of counters which each of the gamesters has, $n+d$ the number of games about which the spectators contend.

Let $a+b$ be raised to the nth power and let the two extreme terms be discarded; let the remainder be multiplied by $a^2+2ab+b^2$ and the extreme terms be rejected; again let there be a multiplication of the remainder by $a^2+2ab+b^2$ and the extremes be rejected, and thus successively let there be as many multiplications as there are units in $\frac{1}{2}d$; and the final product will be the numerator of S's expectation, and the denominator will always be $(a+b)^{n+d}$.

Note that, if d is an odd number, let $d-1$ be substituted for d.

If n is an odd number the numerator and denominator of the expectation can be divided by $a+b$ and the expectation will be made simpler.

EXAMPLE 1. Let 4 be the number of counters which each of the gamesters has, and 10 the number of games about which the spectators contend, and let the skills be equal; the expectation of S is sought.

Hence, $n=4$ and $n+d=10$, therefore $d=6$ and $\frac{1}{2}d=3$. Therefore let $a+b$ be raised to the fourth power, and, always discarding the extremes, let there be 3 multiplications by $a^2+2ab+b^2$:

[p. 256]

$$\begin{array}{l}
a^4| + 4a^3b + 6a^2b^2 + 4ab^3| + b^4 \\
\qquad a^2 + 2ab + b^2 \\
\hline
4a^5b| + 6a^4b^2 + 4a^3b^3 \\
\quad + 8a^4b^2 + 12a^3b^3 + 8a^2b^4 \\
\qquad + 4a^3b^3 + 6a^2b^4| + 4ab^5 \\
\hline
14a^4b^2 + 20a^3b^3 + 14a^2b^4 \\
\qquad a^2 + 2ab + b^2 \\
\hline
14a^6b^2| + 20a^5b^3 + 14a^4b^4 \\
\quad + 28a^5b^3 + 40a^4b^4 + 28a^3b^5 \\
\qquad + 14a^4b^4 + 20a^3b^5| + 14a^2b^6 \\
\hline
48a^5b^3 + 68a^4b^4 + 48a^3b^5 \\
\qquad a^2 + 2ab + b^2 \\
\hline
48a^7b^3| + 68a^6b^4 + 48a^5b^5 \\
\quad + 96a^6b^4 + 136a^5b^5 + 96a^4b^6 \\
\qquad + 48a^5b^5 + 68a^4b^6| + 48a^3b^7 \\
\hline
164a^6b^4 + 232a^5b^5 + 164a^4b^6.
\end{array}$$

And the expectation of S will be equal to

$$(164a^6b^4 + 232a^5b^5 + 164a^4b^6)/(a+b)^{10},$$

and since a and b are equal, the expectation will be

$$\frac{164+232+164}{2^{10}} = \frac{560}{1024} = \frac{35}{64}.$$

EXAMPLE 2. Let 5 be the number of counters which each of the gamesters has, and 10 the number of games about which the spectators contend, so that S denies that the contest will be ended in 10 games; and let the skill of A be to the skill of B as 2 to 1.

Hence, $n=5$ and $n+d=10$, therefore $d=5$. And since d is odd let d be made equal to 4, therefore $\frac{1}{2}d=2$. Therefore let $a+b$ be raised to the fifth power, and if the extremes are always discarded let there be 2 multiplications by $a^2+2ab+b^2$.

[p. 257]

$$
\begin{array}{l}
\qquad\quad a^5|+5a^4b+10a^3b^2+10a^2b^3+5ab^4|+b^5 \\
\qquad\qquad\qquad\qquad a^2+2ab+b^2 \\
\hline
5a^6b|+10a^5b^2+10a^4b^3+\ \ 5a^3b^4 \\
\qquad +10a^5b^2+20a^4b^3+20a^3b^4+10a^2b^5 \\
\qquad\qquad\qquad +\ \ 5a^4b^3+10a^3b^4+10a^2b^5|+5ab^6 \\
\hline
\qquad 20a^5b^2+35a^4b^3+35a^3b^4+20a^2b^5 \\
\qquad\qquad\qquad a^2+2ab+b^2 \\
\hline
20a^7b^2|+35a^6b^3+\ \ 35a^5b^4+\ \ 20a^4b^5 \\
\qquad +40a^6b^3+\ \ 70a^5b^4+\ \ 70a^4b^5+40a^3b^6 \\
\qquad\qquad\qquad +\ \ 20a^5b^4+\ \ 35a^4b^5+35a^3b^6|+20a^2b^7 \\
\hline
\qquad 75a^6b^3+125a^5b^4+125a^4b^5+75a^3b^6.
\end{array}
$$

Therefore the expectation of S will be

$$(75a^6b^3+125a^5b^4+125a^4b^5+75a^3b^6)/(a+b)^9.$$

Or, if the numerator and denominator are divided by $a+b$ since the number n is odd, the expectation will become equal to

$$(75a^5b^3+50a^4b^4+75a^3b^5)/(a+b)^8=25a^3b^3(3a^2+2ab+3b^2)/(a+b)^8.$$

And if we place 2 and 1 for a and b, respectively, the expectation will become $8\times25\times19/6561=3800/6561$.

[p. 258] PROBLEM 21. Let 4 be the number of counters which each of the gamesters has; the ratio of skills is required so that R can claim with equal chance that the contest will be ended within 4 games, and S deny.

Solution. The expectation of S, just found, is $(4a^3b+6a^2b^2+4ab^3)/(a+b)^4$ and since, from the hypothesis, R and S are contending with equal chance then $(4a^3b+6a^2b^2+4ab^3)/(a+b)^4=\frac{1}{2}$ or $a^4-4a^3b-6a^2b^2-4ab^3+b^4=0$. Let $12a^2b^2$ be added to both sides so that $a^4-4a^3b+6a^2b^2-4ab^3+b^4=12a^2b^2$. Hence let the square root be extracted from this which gives $a^2-2ab+b^2=ab\sqrt{(12)}$, or making $a:b::z:1$, $z^2-2z+1=z\sqrt{(12)}$, where the two roots will be found to be 5·274 and 1/5·274. Therefore the ratio of the skill of A to the skill of B would be either as 5·274 to 1 or as 1 to 5·274, for R and S to contend with equal chance.

PROBLEM 22. Let 4 be the number of counters which each of the gamesters has; the ratio of skills is sought, such that R may claim that the contest will be ended within 4 games, S may deny it, and the chances of R and S may be in a given ratio, namely as 3 to 1.

Solution. The expectation of S from 4 games and the given ratio of skills equals $(4a^3b+6a^2b^2+4ab^3)/(a+b)^4$. This expectation, because of the given ratio of chances, is $\frac{1}{4}$. Therefore $(4a^3b+6a^2b^2+4ab^3)/(a+b)^4=\frac{1}{4}$, [p. 259] or $a^4-12a^3b-18a^2b^2-12ab^3+b^4=0$. Now, making $a:b::z:1$, it follows that

$$z^4-12z^3-18z^2-12z+1=0.$$

[In this equation the last term but one has been changed from $12z^3$ to $12z$.] Suppose that this equation is formed from the two quadratics, $z^2+yz+1=0$ and $z^2+pz+1=0$.

Therefore

$$z^4 \begin{matrix}+y\\ \\ +p\end{matrix}\; z^3 \begin{matrix}+py\\ \\ +\;2\end{matrix}\; z^2 \begin{matrix}+y\\ \\ +p\end{matrix}\; z+1=0.$$

Let the coefficients of the homologous terms be compared and it will be seen that $y+p=-12$ and $py+2=-18$ or $py=-20$, whence the equation $y^2+12y=20$ will arise of which the negative root is equal to $-13{\cdot}483$. Let this value be substituted in the place of y and it follows that $z^2-13{\cdot}483z+1=0$, the two roots of which equation will be found to be $13{\cdot}407$ and $1/13{\cdot}407$ nearly; therefore if a is to b either as $13{\cdot}407$ to 1 or as 1 to $13{\cdot}407$, the ratio of chances of R and S will be as 3 to 1.

PROBLEM 23. Let 4 be the number of counters which each of the gamesters has; the ratio of skills is required such that R can with equal chance claim that the contest will be ended within 6 games, and S deny it.

Solution. The expectation of S, for the given number of games and the ratio of skills, equals $(14a^4b^2+20a^3b^3+14a^2b^4)/(a+b)^6$. His expectation, because of the given equality of chances will be equal to $\frac{1}{2}$. Hence, $(14a^4b^2+20a^3b^3+14a^2b^4)/(a+b)^6=\frac{1}{2}$ or $a^6+6a^5b-13a^4b^2-20a^3b^3-13a^2b^4+6ab^5+b^6=0$ and, with $a:b::z:1$, $z^6+6z^5-13z^4-20z^3-13z^2+6z+1=0$.

Suppose that this equation is formed from the following two: [p. 260] $z^2+yz+1=0$ and $z^4+pz^3+qz^2+pz+1=0$.

Therefore

$$\begin{array}{l} z^6+yz^5+z^4 \\ \quad +pz^5+pyz^4+pz^3 \\ \qquad\quad +qz^4+qyz^3+qz^2 \\ \qquad\qquad\quad +pz^3+pyz^2+pz \\ \qquad\qquad\qquad\quad +z^2+yz+1. \end{array}$$

Or

$$z^6 \begin{matrix}+y\\ \\ +p\end{matrix}\; z^5 \begin{matrix}+1\\ +py\\ +q\end{matrix}\; z^4 \begin{matrix}+2p\\ \\ +qy\end{matrix}\; z^3 \begin{matrix}+1\\ +py\\ +q\end{matrix}\; z^2 \begin{matrix}+p\\ \\ +y\end{matrix}\; z+1=0.$$

And, if we compare the coefficients, it follows that $y+p=6$, $1+py+q=-13$, or $py+q=-14$, $2p+qy=-20$. Whence the equation $y^3-6y^2-16y+32=0$ will arise of which one of the roots will be $-2{\cdot}9644$, which being substituted in the place of y in the equation $z^2+yz+1=0$ will give the new equation $z^2-2{\cdot}9644z+1=0$. The two roots will be found to be $2{\cdot}576$ and $1/2{\cdot}576$; therefore if A's skill is to B's skill either as $2{\cdot}576$ to 1 or as 1 to $2{\cdot}576$, R and S will contend with equal chance.

COROLLARY. All equations of this sort, in which the ratio of skills comes to be determined from the counters and the number of games, may always be reduced to dimensions smaller by at least half than is the given number of games; further, the coefficients of the terms equally distant from the extremes will always be the same, and so if it is supposed that those equations are formed from $z^2+yz+1=0$ [the first term has been changed from y^2 to z^2], and another equation of which the coefficients equally distant from the extremes are the same, the comparisons of homologous terms will not be more than is half the number of games, and so the dimensions of quantity y will be smaller by at least half than the dimensions of quantity z.

[p. 261] PROBLEM 24. Supposing the same things as in Problem 20, let A have p counters and B q counters; the expectation of S is sought.

Solution. Take the binomial $a+b$, and, always rejecting the terms in which the dimensions of quantity a exceed the dimensions of quantity b by q, and the terms in which the dimensions of quantity b exceed the dimensions of quantity a by p, let the remaining terms be successively multipled by $a+b$; and let there be as many multiplications as there are units in the given number of games diminished by one, and the numerator of S's expectation will be had of which the denominator will be the power of the binomial $a+b$ designated by the number of games.

EXAMPLE. Let $p=3$ and $q=2$, the number of games 7:

$$\begin{array}{c}
a+b \\
a+b \\
\hline
a^2|+2ab+b^2 \\
a+b \\
\hline
2a^2b+3a^2|+b^3 \\
a+b \\
\hline
2a^3b|+5a^2b^2+3ab^3 \\
a+b \\
\hline
5a^3b^2+8a^2b^3|+3ab^4 \\
a+b \\
\hline
5a^4b^2|+13a^3b^3+8a^2b^4 \\
a+b \\
\hline
13a^4b^3+21a^3b^4|+8a^2b^5.
\end{array}$$

Therefore the expectation of S will be equal to $(13a^4b^3+21a^3b^4)/(a+b)^7$.

[p. 262] PROBLEM 25. Two gamesters, A and B, whose skills are in a given ratio, enter into this agreement, that they should not stop playing before a given number of games has been completed; R and S are two spectators, of whom R claims that, at some time before the contest ends, or as the contest draws to a close, A will have won a certain number of games more than B has won; R's expectation is sought.

Solution. Let n be the number of games that must be completed before A and B may stop playing; let $n-d$ be the number of games about which R and S contend; let the ratio of skills be as a to b. Let $a+b$ be raised to the nth power; then if d is an odd number let as many terms of this power be taken as there are units in $(d+1)/2$; also let as many terms following be taken as already has been taken, but let their coefficients be changed and the coefficients of the preceding terms be prefixed to them in reverse order; but if d is an even number let as many terms of the power $(a+b)^n$ be taken as there are units in $(d+2)/2$; also let as many following terms be taken as there are units in $\frac{1}{2}d$, but let the coefficients of the preceding terms be prefixed to them in reverse order omitting the last of them; and the numerator of R's expectation will be had of which the denominator will be $(a+b)^n$.

EXAMPLE 1. Let 10 be the number of games that must be completed before A and B may stop playing, let 3 be the number of games in which A will at some time defeat B, let the ratio of skills be as 1 to 1. Let $a+b$ be raised to the tenth power, namely

$$a^{10}+10a^9b+45a^8b^2+120a^7b^3+210a^6b^4+252a^5b^5 \\ +210a^4b^6+120a^3b^7+45a^2b^8+10ab^9+b^{10}.$$

[p. 263] (i) $n=10$; (ii) $n-d=3$; therefore $d=7$ and $(d+1)/2=4$. Therefore let 4 terms of that power be taken, namely $a^{10}+10a^9b+45a^8b^2+120a^7b^3$; also let 4 following terms be taken and to these let the coefficients of the preceding terms be prefixed in reverse order; the following terms will come out $120a^6b^4+45a^5b^5+10a^4b^6+a^3b^7$. Therefore R's expectation will be equal to

$$(a^{10}+10a^9b+45a^8b^2+120a^7b^3+120a^6b^4+45a^5b^5+10a^4b^6+1a^3b^7)/(a+b)^{10} \\ =352/1024$$

EXAMPLE 2. Let $n=6$ and $n-d=4$, therefore $d=2$ and $(d+2)/2=2$. Therefore R's expectation will be $(a^6+6a^5b+a^4b^2)/(a+b)^6$.

Note that if d is an odd number the numerator and denominator of R's expectation can be divided by $a+b$.

PROBLEM 26. Two gamesters, A and B, whose skills are in a given ratio, namely as a to b, enter into this agreement: that they should not stop playing before a given number of games has been completed. Present are two spectators R and S, and R claims, S denies, that, at some time before the contest ends, or as the contest draws to a close, A will defeat B in a given number of games q, and also that at some time B will defeat A in a given number of games p; the expectation of R is sought.

Solution. The number of chances by which A may defeat B in a given number of games q may be found through Problem 25.

The number of chances by which B may defeat A in a given number of games p may be found through the same.

Finally, the number of chances by which neither may defeat the other in a given number of games may be found through Problem 24.

Let these cases be added together and from their sum subtract $(a+b)^n$, which gives the numerator of R's expectation, the denominator being $(a+b)^n$.

[p. 264] EXAMPLE. R claims that at some time A will defeat B in 2 games, and also that at some time B will defeat A in 3 games, and the number of games that must be completed is 7.

The number of chances by which A may defeat B in 2 games is

$$a^7+7a^6b+21a^5b^2+21a^4b^3+7a^3b^4+a^2b^5.$$

The number of chances by which B may defeat A in 3 games is

$$1a^4b^3+7a^3b^4+21a^2b^5+7ab^6+b^7.$$

The number of chances by which neither may defeat the other in a given number of games is $13a^4b^3+21a^3b^4$.

The sum of all these chances will be

$$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+22a^2b^5+7ab^6+b^7.$$

Let $(a+b)^7$ be subtracted, or

$$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

The remainder will be $1a^2b^5$. Therefore R's expectation will be $a^2b^5/(a+b)^7$.

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