ON THE

ALGEBRAICAL AND NUMERICAL THEORY OF ERRORS OF OBSERVATIONS AND THE COMBINATION OF OBSERVATIONS.

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PREFACE TO THE FIRST EDITION.

THE Theory of Probabilities is naturally and strongly divided into two parts. One of these relates to those chances which can be altered only by the changes of entire units or integral multiples of units in the fundamental conditions of the problem; as in the instances of the number of dots exhibited by the upper surface of a die, or the numbers of black and white balls to be extracted from a bag. The other relates to those chances which have respect to insensible gradations in the value of the element measured; as in the duration of life, or in the amount of error incident to an astronomical observation.

It may be difficult to commence the investigations proper for the second division of the theory without referring to principles derived from the first. Nevertheless, it is certain that, when the elements of the second division of the theory are established, all reference to the first division is laid aside; and the original connexion is, by the great majority of persons who use the second division, entirely forgotten. The two divisions branch off into totally unconnected subjects; those persons who habitually use one part never have occasion for the other; and practically they become two different sciences.

In order to spare astronomers and observers in natural philosophy the confusion and loss of time which are produced by referring to the ordinary treatises embracing both branches of Probabilities, I have thought it desirable to draw up this tract, relating only to Errors of Observation, and to the rules, derivable from the consideration of these Errors, for the Combination of the Results of Observations. I have thus also the advantage of entering somewhat more fully into several points, of interest to the observer, than can possibly be done in a General Theory of Probabilities.

No novelty, I believe, of fundamental character, will be found in these pages. At the same time I may state that the work has been written without reference to or distinct recollection of any other treatise (excepting only Laplace's *Théorie des Probabilités*); and the methods of treating the different problems may therefore differ in some small degrees from those commonly employed.

G. B. AIRY.

ROYAL OBSERVATORY, GREENWICH,
January 22, 1861.

PREFACE TO THE SECOND EDITION.

The work has been thoroughly revised, but no important alteration has been made: except in the introduction of the new Section 15, and the consequent alteration in the numeration of articles of Sections 16 and 17 (formerly 15 and 16): and in the addition of the Appendix, giving the result of a comparison of the theoretical law of Frequency of Errors with the Frequency actually observed in an extensive series.

G. B. AIRY.

February 20, 1875.

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CORRIGENDA.

Page 47, *dele* line 1, and *substitute* the following:—

Mean Square of Sum of Errors $a + b + c + d +$ &c.

Page 61, between lines 6 and 7, *insert* "final apparent results, as affected by the"

" line 12, *for* 'actual error of' *read* 'apparent'.

" line 14, *for* 'actual errors of the' *read* 'apparent'.

" line 19, *for* 'actual error' *read* 'result'.

ON THE

ALGEBRAICAL AND NUMERICAL THEORY

OF

ERRORS OF OBSERVATIONS

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PART I.

FALLIBLE MEASURES, AND SIMPLE ERRORS OF OBSERVATION.

§ 1. *Nature of the Errors here considered.*

1. THE nature of the Errors of Observation which form the subject of the following Treatise, will perhaps be understood from a comparison of the different kinds of Errors to which different Estimations or Measures are liable.

2. Suppose that a quantity of common nuts are put into a cup, and a person makes an estimate of the number. His estimate may be correct; more probably it will be incorrect. But if incorrect, the error has this

peculiarity, that it is an error of whole nuts. There cannot be an error of a fraction of a nut. This class of errors may be called Errors of Integers. These are not the errors to which this treatise applies.

3. Instead of nuts, suppose water to be put into the cup, and suppose an estimate of the quantity of water to be formed, expressed either by its cubical content, or by its weight. Either of those estimates may be in error by any amount (practically not exceeding a certain limit), proceeding by any gradations of magnitude, however minute. This class of errors may be called Graduated Errors. It is to the consideration of these errors that this treatise is directed.

4. If, instead of nuts or water, the cup be charged with particles of very small dimensions, as grains of fine sand, the state of things will be intermediate between the two considered above. Theoretically, the errors of estimation, however expressed, must be Errors of Integers of Sand-Grains; but practically, these sand-grains may be so small that it is a matter of indifference whether the gradations of error proceed by whole sand-grains or by fractions of a sand-grain. In this case, the errors are practically Graduated Errors.

5. In all these cases, the estimation is of a simple kind; but there are other cases in which the process may be either simple or complex; and, if it is complex, a different class of errors may be introduced. Suppose, for instance, it is desired to know the length of a given road.

A person accustomed to road-measures may estimate its length; this estimation will be subject simply to Graduated Errors. Another person may measure its length by a yard-measure; and this method of measuring, from uncertainties in the adjustments of the successive yards, &c. will also be subject to Graduated Errors. But besides this, it will be subject to the possibility of the omission of registry of entire yards, or the record of too many entire yards; not as a fault of estimate, but as a result of mental confusion. In like manner, when a measure is made with a micrometer; there may be inaccuracy in the observation as represented by the fractional part of the reading; but there may also be error of the number of whole revolutions, or of the whole number of decades of subdivisions, similar to the erroneous records of yards mentioned above, arising from causes totally distinct from those which produce inaccuracy of mere observation. This class of Errors may be called Mistakes. Their distinguishing peculiarity is, that they admit of Conjectural Correction. These Mistakes are not further considered in the present treatise.

6. The errors therefore, to which the subsequent investigations apply, may be considered as characterized by the following conditions:—

They are infinitesimally graduated,

They do not admit of conjectural correction.

7. Observations or measures subject to these errors will be called in this treatise "fallible observations," or "fallible measures."

8. Strictly speaking, we ought, in the expression of our general idea, to use the word "uncertainty" instead of "error." For we cannot at any time assert positively that our estimate or measure, though fallible, is not perfectly correct; and therefore it may happen that there is no "error," in the ordinary sense of the word. And, in like manner, when from the general or abstract idea we proceed to concrete numerical evaluations, we ought, instead of "error," to say "uncertain error;" including, among the uncertainties of value, the possible case that the uncertain error may $= 0$. With this caution, however, in the interpretation of our word, the term "error" may still be used without danger of incorrectness. When the term is qualified, as "Actual Error" or "Probable Error," there is no fear of misinterpretation.

§ 2. *Law of Probability of Errors of any given amount.*

9. In estimating numerically the "probability" that the magnitude of an error will be included between two given limits, we shall adopt the same principle as in the ordinary Theory of Chances. When the numerical value of the "probability" is to be determined *à priori*, we shall consider all the possible combinations which produce error; and the fraction, whose numerator is the number of combinations producing an error which is included between the given limits, and whose denominator is the total number of possible combinations, will be the "probability" that the error will be included between those limits. But when the numerical value is to be deter-

mined from observations, then if the numerator be the number of observations, whose errors fall within the given limits, and if the denominator be the total number of observations, the fraction so formed, when the number of observations is indefinitely great, is the "probability."

10. A very slight contemplation of the nature of errors will lead us to two conclusions :—

First, that, though there is, in any given case, a possibility of errors of a large magnitude, and therefore a possibility that the magnitude of an error may fall between the two values E and $E+\delta e$, where E is large; still it is more probable that the magnitude of an error may fall between the two values e and $e+\delta e$, where e is small; δe being supposed to be the same in both. Thus, in estimating the length of a road, it is less probable that the estimator's error will fall between 100 yards and 101 yards than that it will fall between 10 yards and 11 yards. Or, if the distance is measured with a yard-measure, and mistakes are put out of consideration, it is less likely that the error will fall between 100 inches and 101 inches than that it will fall between 10 inches and 11 inches.

Second, that, according to the accuracy of the methods used and the care bestowed upon them, different values must be assumed for the errors in order to present comparable degrees of probability. Thus, in estimating the road-lengths by eye, an error amounting to 10 yards is sufficiently probable; and the chance that the real error may fall between 10 yards and 11 yards is not contemptibly small. But in measuring by a yard-measure, the probability that the error can amount to 10 yards is so insignificant that no man will think it worth consideration; and the probability that the error may fall between 10 yards and 11 yards will never enter into our thoughts. It may, however, perhaps be judged that an error amounting to 10 inches is about as probable with this kind of measure as an error of 10 yards with eye-estimation; and the probability that the error may fall between 10 inches and 11 inches, with this mode of measuring, may be comparable with the probability of the error, in the rougher estimation, falling between 10 yards and 11 yards.

11. Here then we are led to the idea that the algebraical formula which is to express the probability that an error will fall between the limits e and $e + \delta e$ (where δe is extremely small) will possess the following properties:—

(*A*) Inasmuch as, by multiplying our very narrow interval of limits, we multiply our probability in the same proportion, the formula must be of the form $\phi(e) \times \delta e$.

(*B*) The term $\phi(e)$ must diminish as e increases, and must be indefinitely small when e is indefinitely large.

(*C*) The term $\phi(e)$ must contain a constant symbol or parameter c, which is constant in the expression of the probabilities under the same system of estimation or measure, and is different for different systems of estimation or measure. If (as seems likely), upon taking a proper proportion of magnitudes of error, the law of declension of the probability of errors is the same for delicate measures and for coarse measures, then the formula will be of the form $\psi\left(\frac{e}{c}\right) \times \delta\left(\frac{e}{c}\right)$, or $\psi\left(\frac{e}{c}\right) \times \frac{\delta e}{c}$; where c is small for a delicate system of measures, and large for a coarse system of measures.

[The reader is recommended, in the first instance, to pass over the articles 12 to 21.]

12. Laplace has investigated, by an *à priori* process, well worthy of that great mathematician, the form of the function expressing the law of probability. Without entering into all details, for which we must refer to the *Théorie Analytique des Probabilités*, we may give an idea here of the principal steps of the process.

13. The fundamental principle in this investigation is, that an error, as actually occurring in observation, is not of simple origin, but is produced by the algebraical combination of a great many independent causes of error[1], each of which, according to the chance which affects it independently, may produce an error, of either sign and of different magnitude. These errors are supposed to be of the class of Errors of Integers, which admit of being treated by the usual Theory of Chances; then, supposing the integers to be indefinitely small, and the range of their number to be indefinitely great, the conditions ultimately approach to the state of Graduated Errors.

[1] This is not the language of Laplace, but it appears to be the understanding on which his investigation is most distinctly applicable to single errors of observation.

14. Suppose then that, for one source of error, the errors may be, with equal probability,

$$-n,\ -n+1,\ -n+2, \dots -1,\ 0,\ +1,\ 2, \dots n-2,\ n-1,\ n,$$

the probability of each will be $\dfrac{1}{2n+1}$.

Suppose that, for another source of error, the errors may also be, with equal probability,

$$-n,\ -n+1,\ -n+2, \dots -1,\ 0,\ +1,\ 2, \dots n-2,\ n-1,\ n,$$

and so on for s sources of error. And suppose that we wish to ascertain what is the probability that, upon combining algebraically one error taken from the first series, with one error taken from the second series, and with one error taken from the third series, and so on, we can produce an error l. The first step is, to ascertain how many are the different combinations which will each produce l.

15. Now, if we watch the process of combination, we shall see that the numbers are added by exactly the same law as the addition of indices in the successive multiplications of the polynomial

$$\epsilon^{-n\theta\sqrt{-1}} + \epsilon^{-(n-1)\theta\sqrt{-1}} + \epsilon^{-(n-2)\theta\sqrt{-1}} \dots + \epsilon^{(n-2)\theta\sqrt{-1}} + \epsilon^{(n-1)\theta\sqrt{-1}} + \epsilon^{n\theta\sqrt{-1}},$$

by itself, supposing the operation repeated $s-1$ times. And therefore the number of combinations required will be, the coefficient of $\epsilon^{l\theta\sqrt{-1}}$ (which is also the same as the coefficient of $\epsilon^{-l\theta\sqrt{-1}}$), in the expansion of

$$\left\{\epsilon^{-n\theta\sqrt{-1}} + \epsilon^{-(n-1)\theta\sqrt{-1}} + \epsilon^{-(n-2)\theta\sqrt{-1}} \dots + \epsilon^{(n-2)\theta\sqrt{-1}} + \epsilon^{(n-1)\theta\sqrt{-1}} + \epsilon^{n\theta\sqrt{-1}}\right\}^s.$$

This coefficient will be exhibited as a number uncombined with any power of $\epsilon^{\theta\sqrt{-1}}$, if we multiply the expansion either by $\epsilon^{l\theta\sqrt{-1}}$, or by $\epsilon^{-l\theta\sqrt{-1}}$, or by $\frac{1}{2}(\epsilon^{l\theta\sqrt{-1}}+\epsilon^{-l\theta\sqrt{-1}})$.

The number of combinations required is therefore the same as the term independent of θ in the expansion of

$$\frac{1}{2}(\epsilon^{l\theta\sqrt{-1}}+\epsilon^{-l\theta\sqrt{-1}})\{\epsilon^{-n\theta\sqrt{-1}}+\epsilon^{-(n-1)\theta\sqrt{-1}}+\&\text{c.}+\epsilon^{(n-1)\theta\sqrt{-1}}+\epsilon^{n\theta\sqrt{-1}}\}^s,$$

or the same as the term independent of θ in the expansion of

$$\cos l\theta \times \{1+2\cos\theta+2\cos 2\theta+\ldots\ldots+2\cos n\theta\}^s.$$

And, remarking that if we integrate this quantity with respect to θ, from $\theta=0$ to $\theta=\pi$, the terms depending on θ will entirely disappear, and the term independent of θ will be multiplied by π, it follows that the number of combinations required is the definite integral

$$\frac{1}{\pi}.\int_0^\pi d\theta.\cos l\theta\times\{1+2\cos\theta+2\cos 2\theta\ldots+2\cos n\theta\}^s,$$

$$\text{or } \frac{1}{\pi}.\int_0^\pi d\theta.\cos l\theta\times\left(\frac{\sin\frac{2n+1}{2}\theta}{\sin\frac{1}{2}\theta}\right)^s.$$

And the total number of possible combinations which are, *à priori*, equally probable, is $(2n+1)^s$.

Consequently, the probability that the algebraical combination of errors, one taken from each series, will produce the error l, is

$$\frac{1}{(2n+1)^s} \cdot \frac{1}{\pi} \cdot \int_0^\pi d\theta \,.\, \cos l\theta \times \left(\frac{\sin \frac{2n+1}{2}\theta}{\sin \frac{1}{2}\theta}\right)^s.$$

In subsequent steps, n and s are supposed to be very large.

16. To integrate this, with the kind of approximation which is proper for the circumstances of the case, Laplace assumes

$$\frac{\sin \frac{2n+1}{2}\theta}{(2n+1)\,.\,\sin \frac{1}{2}\theta} = \epsilon^{-\frac{t^2}{s}};$$

(as the exponential is essentially positive, this does not in strictness apply further than $\frac{2n+1}{2}\theta = \pi$; but as succeeding values of the fraction are small, and are raised to the high power s, they may be safely neglected in comparison with the first part of the integral); expanding the sines in powers of θ, and the exponential in powers of $\frac{t^2}{s}$, it will be found that

$$\theta = \frac{t\sqrt{6}}{\sqrt{\{n(n+1)s\}}}\left(1 + \frac{B}{s}t^2 + \&c.\right),$$

where B is a function of n which approaches, as n becomes very large, to the definite numerical value $\frac{1}{10}$. The expression to be integrated then becomes,

$$\frac{1}{\pi}\frac{\sqrt{6}}{\sqrt{\{n(n+1)s\}}}\times$$

$$\int_0^\infty dt.\cos\left[\frac{lt\sqrt{6}}{\sqrt{\{n(n+1)s\}}}\left(1+\frac{B}{s}t^2+\&c.\right)\right].\epsilon^{-t^2}.\left(1+\frac{3B}{s}t^2+\&c.\right).$$

To simplify this integral, it is to be remarked that ϵ^{-t^2} multiplies the whole, and that this factor decreases with extreme rapidity as t increases. While t is small, the terms $\frac{B}{s}t^2$ in the argument of the cosine are unimportant; and when t is large, it matters not whether they are retained or not, because their rejection merely produces a different length of period for the periodical term which is multiplied by an excessively small coefficient. Also it appears (as will be shewn in Article 19) that the integration of such a term as $\cos mt.\epsilon^{-t^2}.3Bt^2$ introduces no infinite term, and therefore when it is divided by the very large number s, this may be rejected. The integral is therefore reduced to this,

$$\frac{1}{\pi}.\frac{\sqrt{6}}{\sqrt{\{n(n+1)s\}}}\int_0^\infty dt.\cos\frac{lt\sqrt{6}}{\sqrt{\{n(n+1)s\}}}.\epsilon^{-t^2}.$$

17. As the first step to this, let us find the value of $\int_0^\infty dt.\epsilon^{-t^2}$. There is no process for this purpose so convenient as the indirect one of ascertaining the solid content of the solid of revolution in which t is the radius of any

section, and z the corresponding ordinate $= \epsilon^{-t^2}$. Let x and y be the other rectangular co-ordinates, so that $t^2 = x^2 + y^2$. Then the solid content may be expressed in either of the following ways:

By polar co-ordinates, solid content

$$= 2\pi . \int_0^\infty dt . t . \epsilon^{-t^2} = \pi.$$

By rectangular co-ordinates, solid content

$$= \int_{-\infty}^\infty dx . \int_{-\infty}^\infty dy . \epsilon^{-(x^2+y^2)} = \int_{-\infty}^\infty dx . \epsilon^{-x^2} . \int_{-\infty}^\infty dy . \epsilon^{-y^2}$$

$$= \left(4\int_0^\infty dx . \epsilon^{-x^2}\right) \times \left(\int_0^\infty dy . \epsilon^{-y^2}\right) = 4\left(\int_0^\infty dt . \epsilon^{-t^2}\right)^2,$$

since, for a definite integral, it is indifferent what symbol be used for the independent variable.

Hence, $$4\left(\int_0^\infty dt . \epsilon^{-t^2}\right)^2 = \pi,$$

and $$\int_0^\infty dt . \epsilon^{-t^2} = \frac{\sqrt{\pi}}{2}.$$

18. Next, to find the value of $\int_0^\infty dt . \cos rt . \epsilon^{-t^2}$. Call this definite integral y. As this is a function of r, it can be differentiated with respect to r; and as the process of integration expressed in the symbol does not apply to r, y can be differentiated by differentiating under the integral sign. Thus

$$\frac{dy}{dr} = -\int_0^\infty dt . t \sin rt . \epsilon^{-t^2}.$$

Integrating by parts, the general integral for $\frac{dy}{dr}$

$$= \frac{1}{2}\sin rt \,.\, \epsilon^{-t^2} - \frac{r}{2}\int dt \,.\, \cos rt \,.\, \epsilon^{-t^2},$$

in which, taking the integral from $t=0$ to $t=\infty$, the first term vanishes, and the second becomes $-\frac{r}{2}y$. Thus we have

$$\frac{dy}{dr} = -\frac{r}{2}y.$$

Integrating this differential equation in the ordinary way,

$$y = C \,.\, \epsilon^{-\frac{r^2}{4}}.$$

Now when $r=0$, we have found by the last article that the value of y for that case is $\frac{\sqrt{\pi}}{2}$. Hence we obtain finally

$$\int_0^\infty dt \,.\, \cos rt \,.\, \epsilon^{-t^2} = \frac{\sqrt{\pi}}{2} \,.\, \epsilon^{-\frac{r^2}{4}}.$$

19. If we differentiate this expression twice with respect to r, we find,

$$\int_0^\infty dt \,.\, t^2 \,.\, \cos rt \,.\, \epsilon^{-t^2} = \sqrt{\pi} \,.\, \left(\frac{1}{4} - \frac{r^2}{8}\right)\epsilon^{-\frac{r^2}{4}};$$

and expressions of similar character if we differentiate four times, six times, &c. The right-hand expressions are never infinite. This is the theorem to which we referred in Article 16, as justifying the rejection of certain terms in the integral.

20. Reverting now to the expression at the end of Article 18, and making the proper changes of notation, we find for the value of the integral at the end of Article 16,

$$\frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{6}}{\sqrt{\{n(n+1)s\}}} \cdot e^{\frac{-6l^2}{4n(n+1)s}}.$$

This expression for the probability that the error, produced by the combination of numerous errors (see Article 14), will be l, is based on the supposition that the changes of magnitude of l proceed by a unit at a time. If now we pass from Errors of Integers to Graduated Errors, we may consider that we have thus obtained all the probabilities that the error will lie between l and $l+1$. In order to obtain all the probabilities that the error will lie between l and $l+\delta l$, we derive the following expression from that above,

$$\frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{6}}{\sqrt{\{4n(n+1) . s\}}} \cdot e^{\frac{-6l^2}{4n(n+1) . s}} . \delta l.$$

Here l is a very large number, expressing the magnitude x of an error which is not strikingly large, by a large multiple of small units.

Let $l = mx$, where m is large; $\delta l = m\delta x$; and the probability that the error falls between x and $x+\delta x$ is

$$\frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{6} . m}{\sqrt{\{4n . (n+1) . s\}}} , e^{\frac{-6m^2}{4n . (n+1) . s} x^2} . \delta x.$$

Let $\frac{4n(n+1) . s}{6m^2} = c^2$, where c may be a quantity of magnitude comparable to the magnitudes which we shall use in applications of the symbol x; then we have finally for the probability that the error will fall between x and $x + \delta x$,

$$\frac{1}{c\sqrt{\pi}} \cdot \epsilon^{-\frac{x^2}{c^2}} \cdot \delta x.$$

This function, it will be remarked, possesses the characters which in Article 11 we have indicated as necessary. We shall hereafter call c the *modulus*.

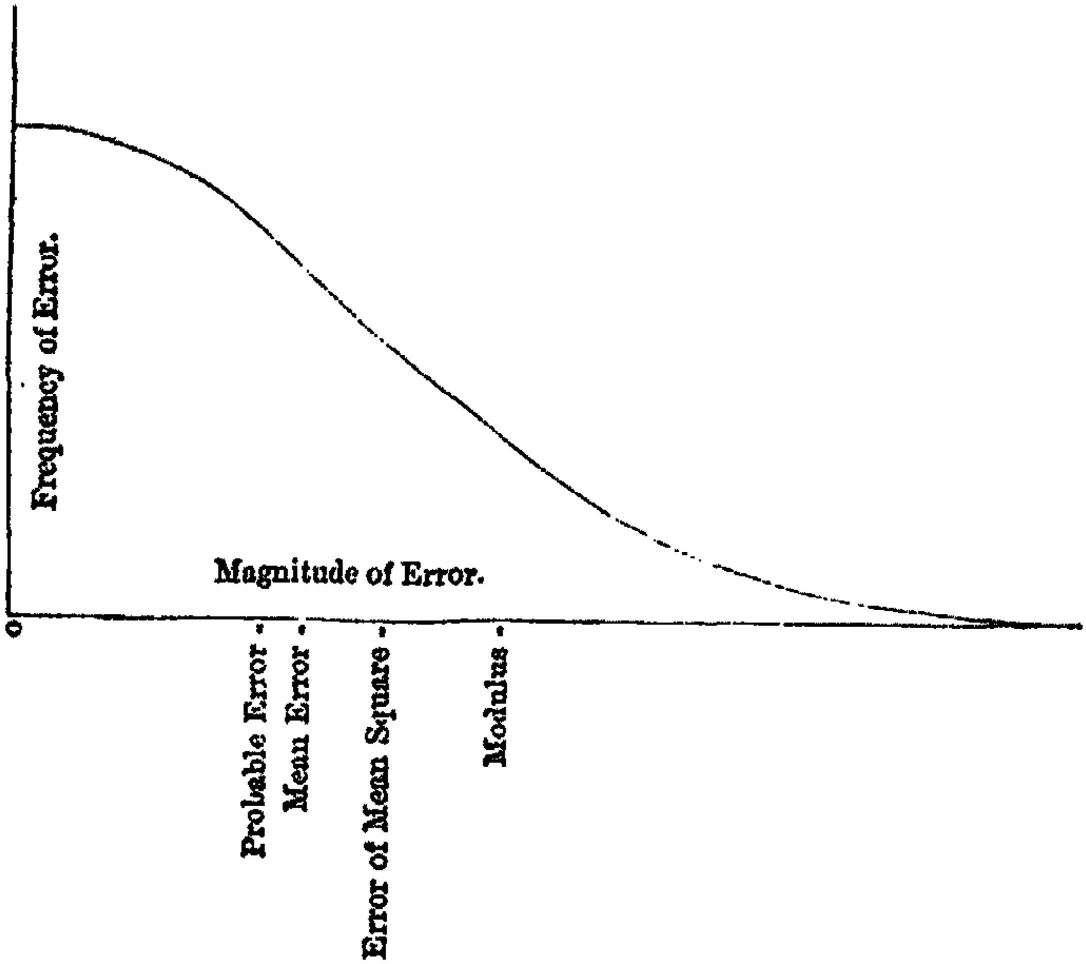
21. Laplace afterwards proceeds to consider the effect of supposing that the probabilities of individual errors, in the different series mentioned in Article 14, are not uniform through each series, as is supposed in Article 14, but vary according to an algebraical law, giving equal probabilities for + or − errors of the same magnitude. And in this case also he finds a result of the same form. For this, however, we refer to the *Théorie Analytique des Probabilités.*

22. Whatever may be thought of the process by which this formula has been obtained, it will scarcely be doubted by any one that the result is entirely in accordance with our general ideas of the frequency of errors. In order to exhibit the numerical law of frequency (that is, the variable factor $\epsilon^{-\frac{x^2}{c^2}}$; which, when multiplied by δx, gives a number proportioned to the probability of errors falling between x and $x + \delta x$), the following table is computed;

TABLE OF VALUES OF $\epsilon^{-\frac{x^2}{c^2}}$.

$\frac{x}{c}$	$\epsilon^{-\frac{x^2}{c^2}}$	$\frac{x}{c}$	$\epsilon^{-\frac{x^2}{c^2}}$
0·0	1·0000	2·6	0·001159
0·1	0·9901	2·7	0·0006823
0·2	0·9608	2·8	0·0003937
0·3	0·9139	2·9	0·0002226
0·4	0·8521	3·0	0·0001234
0·5	0·7788	3·1	0·00006706
0·6	0·6977	3·2	0·00003571
0·7	0·6126	3·3	0·00001864
0·8	0·5273	3·4	0·000009540
0·9	0·4449	3·5	0·000004785
1·0	0·3679	3·6	0·000002353
1·1	0·2982	3·7	0·000001134
1·2	0·2369	3·8	0·0000005355
1·3	0·1845	3·9	0·0000002480
1·4	0·1409	4·0	0·0000001125
1·5	0·1054	4·1	0·00000005006
1·6	0·07731	4·2	0·00000002183
1·7	0·05558	4·3	0·000000009330
1·8	0·03916	4·4	0·000000003909
1·9	0·02705	4·5	0·000000001605
2·0	0·01832	4·6	0·0000000006461
2·1	0·01216	4·7	0·0000000002549
2·2	0·007907	4·8	0·00000000009860
2·3	0·005042	4·9	0·00000000003738
2·4	0·003151	5·0	0·00000000001389
2·5	0·001930		

23. And to present more clearly to the eye the import of these numbers, the following curve is constructed, in which the abscissa represents $\frac{x}{c}$, or the proportion of the magnitude of an error to the modulus, and the ordinate represents the corresponding frequency of errors of that magnitude.



Here it will be remarked that the curve approaches the abscissa by an almost uniform descent from Magnitude of Error = 0 to Magnitude of Error = $1{\cdot}7 \times$ Modulus; and that after the Magnitude of Error amounts to $2{\cdot}0 \times$ Modulus, the Frequency of Error becomes practically insensible. This is precisely the kind of law which we should *à priori* have expected the Frequency of Error to follow; and which, without such an investigation as Laplace's, we might have assumed generally; and for which, having assumed a general form, we might have searched an algebraical law. For these reasons, we shall, through the rest of this treatise, assume the law of frequency

$$\frac{1}{c\sqrt{\pi}}\epsilon^{-\frac{x^2}{c^2}}.\delta x,$$

as expressing the probability of errors occurring with magnitude included between x and $x+\delta x$.

§ 3. *Consequences of the Law of Probability or Frequency of Errors, as applied to One System of Measures of One Element.*

24. The Law of Probability of Errors or Frequency of Errors, which we have found, amounts practically to this. Suppose the total number of Measures to be A, A being a very large number; then we may expect the number of errors, whose magnitudes fall between x and $x+\delta x$, to be

$$\frac{A}{c\sqrt{\pi}}\epsilon^{-\frac{x^2}{c^2}}.\delta x,$$

where c is a modulus, constant for One System of Measures, but different for Different Systems of Measures. It is partly the object of the following investigations to give the means of determining either the modulus c, or other constants related to it, in any given system of practical errors.

25. This may be a convenient opportunity for remarking expressly that the fundamental suppositions of La-

place's investigation, Article 14, assume that the law of Probability of Errors applies equally to positive and to negative errors. It follows therefore that the formula in Article 24 must be received as applying equally to positive and to negative errors. The number A includes the whole of the measures, whether their errors may happen to be positive or negative.

26. Conceive now that the true value of the Element which is to be measured is known (we shall hereafter consider the more usual case when it is not known), and that the error of every individual measure can therefore be found. The readiest method of inferring from these a number which is closely related to the Modulus is, to take the mean of all the positive errors without sign, and to take the mean of all the negative errors without sign (which two means, when the number of observations is very great, ought not to differ sensibly), and to take the numerical mean of the two. This may be called the Mean Error. It is to be regarded as a mere numerical quantity, without sign. Its relation to the Modulus is thus found. Since the number of errors whose magnitude is included between x and $x + \delta x$ is $\frac{A}{c\sqrt{\pi}} \epsilon^{-\frac{x^2}{c^2}} . \delta x$, and the magnitude of each error does not differ sensibly from x, the sum of these errors will be sensibly $\frac{A}{c\sqrt{\pi}} \epsilon^{-\frac{x^2}{c^2}} . x\delta x$; and the sum of all the errors of positive sign will be

$$\frac{A}{c\sqrt{\pi}} \int_0^\infty dx \,.\, \epsilon^{-\frac{x^2}{c^2}} . x = \frac{cA}{2\sqrt{\pi}}.$$

The number of errors of positive sign is

$$\frac{A}{c\sqrt{\pi}}\int_0^\infty dx\,.\,\epsilon^{-\frac{x^2}{c^2}} = \frac{A}{2}\,.$$

Dividing the preceding expression by this,

$$\text{Mean positive error} = \frac{c}{\sqrt{\pi}}\,.$$

Similarly,

$$\text{Mean negative error} = \frac{c}{\sqrt{\pi}}\,.$$

And therefore,

$$\text{Mean Error} = \frac{c}{\sqrt{\pi}} = c \times 0{\cdot}564189.$$

And conversely,

$$c = \text{Mean Error} \times 1{\cdot}772454.$$

By this formula, c can be found with ease when the series of errors is exhibited.

27. It is however sometimes convenient (as will appear hereafter, Article 61) to use a method of deduction derived from the Squares of Errors. The positive and negative errors are then included under the same formula. If we form the mean of the squares, and extract the square root of that mean, we may appropriately call it the Error of Mean Square. This, like the Mean Error, is a numerical quantity, without sign. To investigate it in terms of c, we remark that the sum of the squares of errors between x and $x + \delta x$ (formed as in the last Article) will be

$$\frac{A}{c\sqrt{\pi}}\epsilon^{-\frac{x^2}{c^2}}\,.\,\delta x \times x^2$$

and the sum of all the squares of errors will be

$$\frac{A}{c\sqrt{\pi}}\int_{-\infty}^{+\infty} dx \,.\, \epsilon^{-\frac{x^2}{c^2}} \,.\, x^2 = \Big|_{-\infty}^{+\infty}\left\{\frac{-Ac}{2\sqrt{\pi}} x \,.\, \epsilon^{-\frac{x^2}{c^2}}\right\}$$
$$+ \frac{Ac}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} dx \,.\, \epsilon^{-\frac{x^2}{c^2}}.$$

The first term vanishes between the limits $-\infty$ and $+\infty$, and the second term $= +\frac{Ac^2}{2}$. The whole number of errors is A. Hence the Mean Square is $\frac{c^2}{2}$, and the Error of Mean Square is

$$c\sqrt{\frac{1}{2}} = c \times 0{\cdot}707107;$$

$$\text{or } c = \text{Error of Mean Square} \times 1{\cdot}414214.$$

28. It has however been customary to make use of a different number, called the Probable Error. It is not meant by this term that the number used is a more probable value of error than any other value, but that, when the positive sign is attached to it, the number of positive errors larger than that value is about as great as the number of positive errors smaller than that value: and that, when the negative sign is attached to it, the same remark applies to the negative errors. The Probable Error itself is a numerical quantity, without sign. To ascertain the algebraical condition which this requires, we have only to remark that, as the number of positive errors up to the value x is $\frac{A}{c\sqrt{\pi}}\int_0^x dx \,.\, \epsilon^{-\frac{x^2}{c^2}}$, and as the whole number of

positive errors is $\frac{A}{2}$, and half the whole number of positive errors is $\frac{A}{4}$, we must find the value of x which makes

$$\frac{1}{c\sqrt{\pi}}\int_0^x dx \,.\, \epsilon^{-\frac{x^2}{c^2}}, \text{ or } \frac{1}{\sqrt{\pi}}\int_0^w dw \,.\, \epsilon^{-w^2}, \text{ equal to } \frac{1}{4}.$$

29. For this purpose, we must be prepared with a table of the numerical values of $\frac{1}{\sqrt{\pi}}\int_0^w dw \,.\, \epsilon^{-w^2}$. It is not our business to describe here the process by which the numerical values are obtained (and which is common to the integrals of all expressible functions); we shall merely give the following table, which is abstracted from tables in Kramp's *Refractions* and in the Encyclopædia Metropolitana, Article *Theory of Probabilities*.

TABLE OF THE VALUES OF $\frac{1}{\sqrt{\pi}}\int_0^w dw \,.\, \epsilon^{-w^2}$.

w	Integral.	w	Integral.	w	Integral.
0·0	0·000000	1·1	0·440103	2·2	0·499068
0·1	0·056232	1·2	0·455157	2·3	0·499428
0·2	0·111351	1·3	0·467004	2·4	0·499655
0·3	0·164313	1·4	0·476143	2·5	0·499796
0·4	0·214196	1·5	0·483053	2·6	0·499881
0·5	0·260250	1·6	0·488174	2·7	0·499932
0·6	0·301928	1·7	0·491895	2·8	0·499962
0·7	0·338901	1·8	0·494545	2·9	0·499979
0·8	0·371051	1·9	0·496395	3·0	0·499988
0·9	0·398454	2·0	0·497661		
1·0	0·421350	2·1	0·498510	∞	0·500000

By interpolation among these, we find that the value of w which gives for the value of the integral 0·25, is 0·476948; or the Probable Error, which is the corresponding value of x, is $c \times 0·476948$. And, conversely, c = Probable Error × 2·096665.

30. The reader will advantageously remark in this table how nearly all the errors are included within a small value of w or $\frac{x}{c}$. For it will be remembered that the Integral when multiplied by A (the entire number of positive and negative errors) expresses the number of errors up to that value of w or $\frac{\text{error}}{c}$. Thus it appears that from $w=0$ up to $w=1·65$ or $\frac{\text{error}}{c}=1·65$, we have already obtained $\frac{49}{50}$ of the whole number of errors of the same sign; and from $w=0$ up to $w=3·0$, we have obtained $\frac{49999}{50000}$ of the whole number of errors of the same sign.

31. Returning now to the results of the investigations in Articles 26, 27, 28, 29; we may conveniently exhibit the relations between the values of the different constants therein found, by the following table:—

PROPORTIONS OF THE DIFFERENT CONSTANTS.

	Modulus.	Mean Error.	Error of Mean Square.	Probable Error.
In terms of Modulus ...	1·000000	0·564189	0·707107	0·476948
In terms of Mean Error	1·772454	1·000000	1·253314	0·845369
In terms of Error of Mean Square	1·414214	0·797885	1·000000	0·674506
In terms of Probable Error	2·096665	1·182916	1·482567	1·000000

32. To distinguish each of the errors, really occurring in observations, from the "Mean Error," "Error of Mean Square," "Probable Error," which are mere numerical deductions made according to laws framed for convenience only, we shall usually designate an error really occurring (whether its magnitude be known or not) by the term "Actual Error."

§ 4. *Remarks on the application of these processes in particular cases.*

33. It must always be borne in mind that the law of frequency of errors does not exactly hold except the number of errors is indefinitely great. With a limited number of errors, the law will be imperfectly followed; and the deductions, made on the supposition that the law is strictly followed, will be or may be inaccurate or inconsistent.

Thus, if we investigate the value of the modulus, first by means of the Mean Error, secondly by the Error of Mean Square, we shall probably obtain discordant results. We cannot assert *à priori* which of these is the better.

34. There is one case which occurs in practice so frequently that it deserves especial notice. In collecting the results of a number of observations, it will frequently be found that, while the results of the greater number of observations are very accordant, the result of some one single observation gives a discordance of large magnitude. There is, under these circumstances, a strong temptation to erase the discordant observation, as having been manifestly affected by some extraordinary cause of error. Yet a consideration of the law of Frequency of Error, as exhibited in the last Section (which recognizes the possible existence of large errors), or a consideration of the formation of a complex error by the addition of numerous simple errors, as in Article 14 (which permits a great number of simple errors bearing the same sign to be aggregated by addition of magnitude, and thereby to produce a large complex error), will shew that such large errors may fairly occur; and if so, they must be retained. We may perhaps think that where a cause of unfair error *may* exist (as in omission of clamping a zenith-distance-circle), and where we know by certain evidence that in some instances that unfair cause *has* actually come into play, there is sufficient reason to presume that it has come into play in an instance before us. Such an explanation, however, can only be admitted with the utmost caution.