Bayes Theorem - Example

Recall that Bayes Theorem has both a discrete and continuous form. We have seen the continuous form, here is the general discrete form:

$$P(B_k|A) = \frac{P(B_k) \times P(A|B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A|B_i)}, \ k = 1, 2, \ldots, n.$$

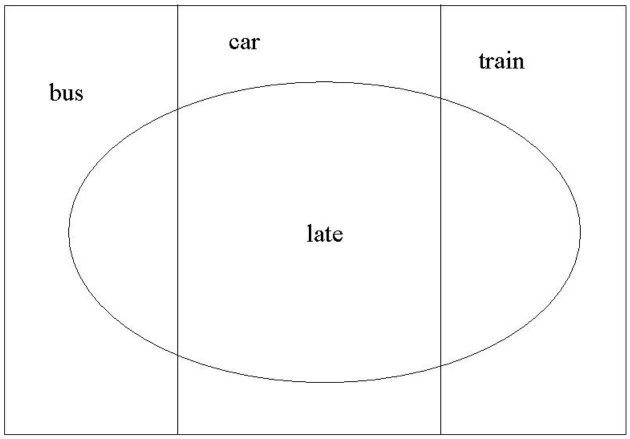
Here is a simple example using it:

Suppose that Bob can decide to go to work by one of three modes of transportation, car, bus, or commuter train. Because of high traffic, if he decides to go by car, there is a 50% chance he will be late. If he goes by bus, which has special reserved lanes but is sometimes overcrowded, the probability of being late is only 20%. The commuter train is almost never late, with a probability of only 1%, but is more expensive than the bus.

(a) Suppose that Bob is late one day, and his boss wishes to estimate the probability that he drove to work that day by car. Since he does not know which mode of transportation Bob usually uses, he gives a prior probability of $\frac{1}{3}$ to each of the three possibilities. What is the boss' estimate of the probability that Bob drove to work?

(b) Suppose that a coworker of Bob's knows that he almost always takes the commuter train to work, never takes the bus, but sometimes, 10% of the time, takes the car. What is the coworkers probability that Bob drove to work that day, given that he was late?

Solution: The Venn diagram would be:



(a) We have the following information given in the problem:

$$\begin{aligned} Pr\{\text{ bus }\} = Pr\{\text{ car }\} = Pr\{\text{ train }\} &= \frac{1}{3} \\ Pr\{\text{ late } \mid \text{ car }\} &= 0.5 \\ Pr\{\text{ late } \mid \text{ train }\} &= 0.01 \\ Pr\{\text{ late } \mid \text{ bus }\} &= 0.2 \end{aligned}$$

We want to calculate $Pr\{\text{ car } \mid \text{ late }\}$.

By Bayes Theorem, this is

$Pr\{\text{ car } \mid \text{ late }\}$

$$= \frac{Pr\{\text{ late } \mid \text{ car }\}Pr\{\text{ car }\}}{Pr\{\text{ late } \mid \text{ car }\}Pr\{\text{ car }\} + Pr\{\text{ late } \mid \text{ bus }\}Pr\{\text{ bus }\} + Pr\{\text{ late } \mid \text{ train }\}Pr\{\text{ train }\}}$$

$$= \frac{0.5 \times 1/3}{0.5 \times 1/3 + 0.2 \times 1/3 + 0.01 \times 1/3}$$

$$= 0.7042$$

(b) Repeat the identical calculations as the above, but instead of the prior probabilities being $\frac{1}{3}$, we use $Pr\{\text{ bus}\} = 0$, $Pr\{\text{car}\} = 0.1$, and $Pr\{\text{ train }\} = 0.9$. Plugging in to the same equation with these three changes, we get $Pr\{\text{ car } \mid \text{ late }\} = 0.8475$