Inference for μ : A&B Ch 7.1 ; Colton Ch 4;

CI(μ): small n: => "Student" 's t distribution

Use when we replace σ by s, an estimate of σ in CI's and tests .

(1) Assume that either
(a) the Y values are either normally distributed or
(b) if not, n is large enough so that the Central Limit Theorem guarantees that the distribution of possible $\bar{y}$'s is well enough approximated by a Gaussian distrn.

(2) Choose the desired degree of confidence [50%, 80%, 90%, 99...] as before.

(3) Proceed as above, except that use t Distribution rather than Z -- find the t value such that xx% of the distribution is between -t and +t. The cutpoints for %-iles of the t distribution vary with the amount of data used to estimate σ.

"Student"'s 't distribution is (conceptual) distribution one gets if...

- take samples (of given size n) from Normal(μ,σ) distribution
- form the quantity $t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$ from each sample
- compile a histogram of the results

or, in Gossett's own words ...(W.S. Gossett 1908)

> *"Before I had succeeded in solving my problem analytically, I had endeavoured to do so empirically [i.e. by simulation]. The material I used was a ... table containing the height and left middle finger measurements of 3000 criminals.... The measurements were written out on 3000 pieces of cardboard, which were then very thoroughly shuffled and drawn at random... each consecutive set of 4 was taken as a sample... [i.e. n=4 above]... and the mean [and] standard deviation of each sample determined.... This provides us with two sets of... 750 z's on which to test the theoretical results arrived at. The height and left middle finger... table was chosen because the distribution of both was approximately normal..."*

Student"'s 't distribution (continued)

Distribution (histogram of sampling distribution)

- is symmetric around 0 (just like $Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$)
- has a shape like that of the Z distribution, but with SD slightly larger than unity i.e. slightly flatter & more wide-tailed; $Var(t) = \frac{df}{df-2}$
- shape becomes indistinguishable from Z distribution as n -> ∞ (in fact as n goes much beyond 30)
- Instead of $\pm 1.96 \frac{\sigma}{\sqrt{n}}$ to enclose μ with 95% confidence, we need
  - $\pm 3.182 \frac{s}{\sqrt{n}}$ if n = 4 (3 degrees of freedom or 'df')
  - $\pm 2.228 \frac{s}{\sqrt{n}}$ if n = 11 (10 df)
  - $\pm 2.086 \frac{s}{\sqrt{n}}$ if n = 21 (20 df)
  - $\pm 1.980 \frac{s}{\sqrt{n}}$ if n = 121 (120 df)
  - $\pm 1.96 \frac{s}{\sqrt{n}}$ if n = ∞ (∞ df)

• Test of $\mu = \mu_0$	CI for μ
$Ratio = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$	$\bar{x} \pm t \frac{s}{\sqrt{n}}$

WORKED EXAMPLE : CI and Test of Significance

Response of interest: D: INCREASE IN HOURS OF SLEEP with DRUG

Test: $H_0: \mu_D = 0$ vs $H_{alt}: \mu_D \neq 0$
$\alpha = 0.05$ (2-sided);

Data:

Subject	HOURS of SLEEP† DRUG	PLACEBO	DIFFERENCE Drug - Placebo
1	6.1	5.2	0.9
2	7.0	7.9	-0.9
3	8.2	3.9	4.3
4	.	.	2.9
5	.	.	1.2
6	.	.	3.0
7	.	.	2.7
8	.	.	0.6
9	.	.	3.6
10	.	.	-0.5
			$\bar{d} = 1.78$
SD of 10 differences			SD[d] = 1.77

Test statistic $= \dfrac{1.78 - [0]}{\frac{1.77}{\sqrt{10}}} = 3.18$ CR:ref|t_9|=2.26

Since 3.18 > 2.26, "Reject" H_0

95% CI for μ_D

$= 1.78 \pm t_9 \dfrac{1.77}{\sqrt{10}} = 1.78 \pm 1.26 = 0.5$ to 3.0 hours

NOTE: I deliberately omitted the full data on the drug and placebo conditions: all we need for the analysis are the 10 differences.

What if not sure d's come from a Gaussian Distribution?

[for t: Gaussian data or (via CLT) Gaussian statistic ($\bar{d}$)

WORKED EXAMPLE *C P G Barker* The Lancet Vol 345 . April 22, 1995, p 1047.

Posture, blood flow, and prophylaxis of venous thromboembolism

Sir--Ashby and colleagues (Feb 18, p 419) report adverse effects of posture on femoral venous blood flow. They noted a moderate reduction velocity when a patient was sitting propped up at 35° in a hospital bed posture and a further pronounced reduction when the patient was sitting with legs dependent. Patients recovering from operations are often asked to sit in a chair with their feet elevated on a footrest. The footrests used in most hospitals, while raising the feet, compress the posterior aspect of the calf. Such compression may be important in the aetiology of venous thrombo-embolism. We investigated the effect of a footrest on blood flow in the deep veins of the calf by dynamic radionuclide venography.

Calf venous blood flow was measured in fifteen young (18-31 years) healthy male volunteers. 88 MBq technetium-99m-labelled pertechnetate in 1 mL saline was injected into the lateral dorsal vein of each foot, with ankle tourniquets inflated to 40 mm Hg, and the time the bolus took to reach the lower border of the patella was measured (Sophy DSX Rectangular Gamma Camera). Each subject had one foot elevated with the calf resting on the footrest and the other plantegrade on the floor as a control. *The mean transit time of the bolus to the knee was 24.6 s (SE 2.2) for elevated feet and 14.8 s (SE 2.2) for control feet [see figure overleaf]. The mean delay was 9.9 s (95% CI 7.8–12.0).*

Simple leg elevation without hip flexion increases leg venous drainage and femoral venous blood flow. The footrest used in this study raises the foot by extension at the knee with no change in the hip position. Ashby and colleagues' findings suggest that such elevation without calf compression would produce an increase in blood flow. Direct pressure of the posterior aspect of the calf therefore seems to be the most likely reason for the reduction in flow we observed. Sitting cross-legged also reduced calf venous blood flow, probably by a similar mechanism. If venous stasis is important in the aetiology of venous thrombosis, the practice of nursing patients with their feet elevated on footrests may need to be reviewed.

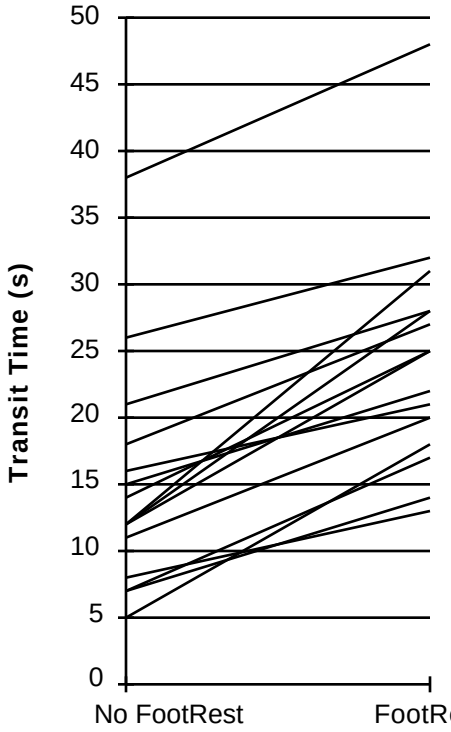
JH's Analysis of raw data [data abstracted by eye, so my calculations won't match exactly with those in text]

$\bar{d}$(SD) = 9.8(4.1); $t = \dfrac{9.8 - [0]}{4.1/\sqrt{15}} = \dfrac{9.8}{1.0} = 9.8 > t_{14,0.05}$ of 2.145

difference is 'off the t-scale'

95% CI on μ_D: $9.8 \pm 2.145[1.0] = 7.7$ to 11.9 s

WORKED EXAMPLE: Leg Elevation (continued)



	No FootRest	FootRest	Delay
	38	48	10
	26	32	6
	21	28	7
	18	27	9
	16	21	5
	15	22	7
	14	25	11
	12	28	16
	12	31	19
	12	25	13
	11	20	9
	8	13	5
	7	17	10
	7	14	7
	5	18	13
mean	14.8	24.6	9.8
SD	8.5	8.7	4.1
SEM	2.2	2.2	1.0

Remarks:

Whereas mean of 15 differences between 2 conditions is arithmetically equal to the difference of the 2 means of 15, the SE of the mean of these 15 differences is not the same as the SE of the difference of two independent means.

Var(diff in means) = Var(mean1) + Var(mean2) – 2Covariance(mean1,mean2)

Authors continue to report the SE of each of the 2 means, but they are of little use here, since we are not interested in the means per se, but in the mean difference.

Calculating Var(mean difference) = Var(diff in means) = Var(mean1) + Var(mean2) assumes that we used one set of 15 subjects for the No FootRest condition, and a different set of 15 for the FootRest condition, a much noisier contrast. As it is, even this inefficient analysis would have sufficed here because the 'signal' was so much greater than the 'noise'

See article On Reserve on display of data from pairs.

Sample Size for CI's and test involving μ

n to yield (2-sided) CI with margin of error m at confidence level 1- α (see M&M p 438)

|--- margin of error --- >|
(----------•----------)

• large-sample CI: $\bar{x} \pm Z_{\alpha/2}\, SE(\bar{x}) = \bar{x} \pm m$

• $SE(\bar{x}) = \sigma / \sqrt{n}$, so...

$$n = \frac{\sigma^2 \cdot Z_{\alpha/2}^2}{m^2}$$

If n small, replace $Z_{\alpha/2}$ by $t_{\alpha/2}$

Typically, won't know σ so use guesstimate;

In planning n for example just discussed, authors might have had pilot data on inter leg differences in transit time -- with both legs in the No FootRest position. Sometimes, one has to 'ask around' as to what the SD of the d's will be. Always safer to assume a higher SD than might turn out to be the case.