Denote the "true" (but unobservable) values by X and Y and the observed (error-containing) measurements by X' and Y'. We quantify the degree to which the errors in X' and Y' distort the correlation $\rho_{X,Y}$ and the slope $\beta_{Y/X}$

From the general formulae

$$\rho_{X,Y} = \frac{E[XY] - E[X] \times E[Y]}{SD[X] \times SD[Y]} = \frac{Covar[X,Y]}{\sqrt{Var[X] \times Var[Y]}} \quad (1)$$

and

$$\beta_{Y/X} = \frac{E[XY] - E[X]E[Y]}{VAR[X]} = \rho_{X,Y} \frac{SD[Y]}{SD[X]}, \quad (2)$$

we can derive the consequences of the errors in X and in Y.

Let $X' = X + \epsilon_X$ where ϵ_X has mean 0 and variance $Var[\epsilon_X]$, and is independent of X, so that

$$E[X'] = E[X + \epsilon_X] = E[X] + E[\epsilon_X] = E[X] + 0 = E[X] \quad (3)$$

$$Var[X'] = Var[X + \epsilon_X] = Var[X] + Var[\epsilon_X] \quad (4)$$

Let $Y' = Y + \epsilon_Y$, where ϵ_Y has mean 0 and variance $Var[\epsilon_Y]$, and is independent of Y, so that

$$E[Y'] = E[Y + \epsilon_Y] = E[Y] + E[\epsilon_Y] = E[Y] + 0 = E[Y] \quad (5)$$

$$Var[Y'] = Var[Y + \epsilon_Y] = Var[Y] + Var[\epsilon_Y] \quad (6)$$

$$E[X'Y'] = E[\{X + \epsilon_X\}\{Y + \epsilon_Y\}] = E[XY + \epsilon_X Y + \epsilon_Y X + \epsilon_X \epsilon_Y] = E[XY]. \quad (7)$$

By definition

$$ICC[X] = \frac{Var[X]}{Var[X] + Var[\epsilon_X]} \;\&\; ICC[Y] = \frac{Var[Y]}{Var[Y] + Var[\epsilon_Y]}, \quad (8)$$

with

$$0 \le ICC[X] \le 1 \;\&\; 0 \le ICC[Y] \le 1. \quad (9)$$

1 $\rho_{X',Y'}$: expected correlation of two error-containing variables

$$\begin{aligned}\rho_{X',Y'} &= \frac{E[X'Y'] - E[X'] \times E[Y']}{SD[X'] \times SD[Y']} \\ &= \frac{E[XY] - E[X] \times E[Y]}{\sqrt{Var[X'] \times Var[Y']}} \\ &= \frac{Covar[X,Y]}{\sqrt{\{Var[X'] + Var[\epsilon_X]\} \times \{Var[Y'] + Var[\epsilon_Y]\}}}\end{aligned}$$

dividing above and below by $\sqrt{Var[X] \times Var[Y]}$

$$\begin{aligned} &= \frac{\frac{Cov[X,Y]}{\sqrt{Var[X] \times Var[Y]}}}{\sqrt{\frac{Var[X'] + Var[\epsilon_X]}{Var[X]} \times \frac{Var[Y'] + Var[\epsilon_Y]}{Var[Y]}}} \\ &= \frac{\rho_{X,Y}}{\sqrt{\frac{1}{ICC[X]} \times \frac{1}{ICC[Y]}}}\end{aligned}$$

so that...

$$\boxed{\rho_{X',Y'} = \sqrt{ICC[X]} \times \sqrt{ICC[Y]} \times \rho_{X,Y} \le \rho_{X,Y}.}$$

Thus, the correlation is **attenuated*** (dampened/weakened) by the imperfections (random errors) in the X and Y measurements.

One can reverse the equation to get a **"de-attenuated" correlation**:

$$\boxed{\rho_{X,Y} = \frac{\beta_{X',Y'}}{\sqrt{ICC[X]} \times \sqrt{ICC[Y]}}}$$

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http://www.m-w.com/dictionary/attenuate
*Main Entry:  1attenuate ; Function:  adjective
Etymology:  Middle English attenuat, from Latin attenuatus, past
participle of attenuare to make thin, from ad- + tenuis thin
1 :  reduced especially in thickness, density, or force
2 :  tapering gradually usually to a long slender point
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2 $\beta_{Y'/X'}$: expected slope of error-containing Y on error-containing X

$$\begin{aligned}
\beta_{Y'/X'} &= \frac{E[X'Y'] - E[X'] \times E[Y']}{Var[X']} \\
&= \frac{E[XY] - E[X] \times E[Y]}{Var[X'] + Var[\epsilon_X]} \\
&= \frac{Covar[X,Y]}{Var[X'] + Var[\epsilon_X]} \\
&= \frac{Covar[X,Y]}{Var[X]} \times \frac{Var[X]}{Var[X'] + Var[\epsilon_X]} \\
&= \beta_{Y/X} \times ICC[X].
\end{aligned}$$

so that...

$$\boxed{\beta_{Y'/X'} = \beta_{Y/X} \times ICC[X] \leq \beta_{Y/X}}$$

i.e., the slope is **attenuated** (dampened / weakened / flattened / moved towards 0) by the imperfections in the X measurements. Random errors in Y add to the residual variation, and thus increase the instability of the estimated slope, but do not (on average) attenuate the slope.

One can reverse the equation to get a **"de-attenuated" slope**:

$$\boxed{\beta_{Y'/X'} = \frac{\beta_{Y'/X'}}{ICC[X]}}$$

3 Relationship between test-retest (X', X'') and ICC[X]

X' and X'' denote 2 independent measurements of the X on a randomly selected individual, e.g., measuring one's cholesterol / height/ IQ twice in a short period of time, where X has not changed, and where ϵ_1 and ϵ_2 are independent.

In psychometrics, the term *"test-retest"* is reserved for a self-administered test, such as a questionnaire that is completed by the subject rather than an observer or test-administrator. Otherwise (e.g., if one wishes to study intra-observer or inter-observer variation) psychometricians speak of observer variation, rather than test-retest, studies.

Exercise: Show that

$$\boxed{\rho_{X',X''} = ICC[X]}$$

4 Relationship between $\rho_{X',X}$ and ICC[X]

This applies when we can think of X as the 'gold standard'.

Exercise: Show that

$$\boxed{\rho_{X',X} = \sqrt{ICC[X]}}$$