Significance - Statistics making sense

Bernoulli and the Foundations of Statistics. Can you correct a 300-year-old error?

Julian Champkin

Ars Conjectandi is not a book that non-statisticians will have heard of, nor one that many statisticians will have heard of either. The title means 'The Art of Conjecturing' – which in turn means roughly 'What You Can Work Out From the Evidence.' But it is worth statisticians celebrating it, because it is the book that gave an adequate mathematical foundation to their discipline, and it was published 300 years ago this year.

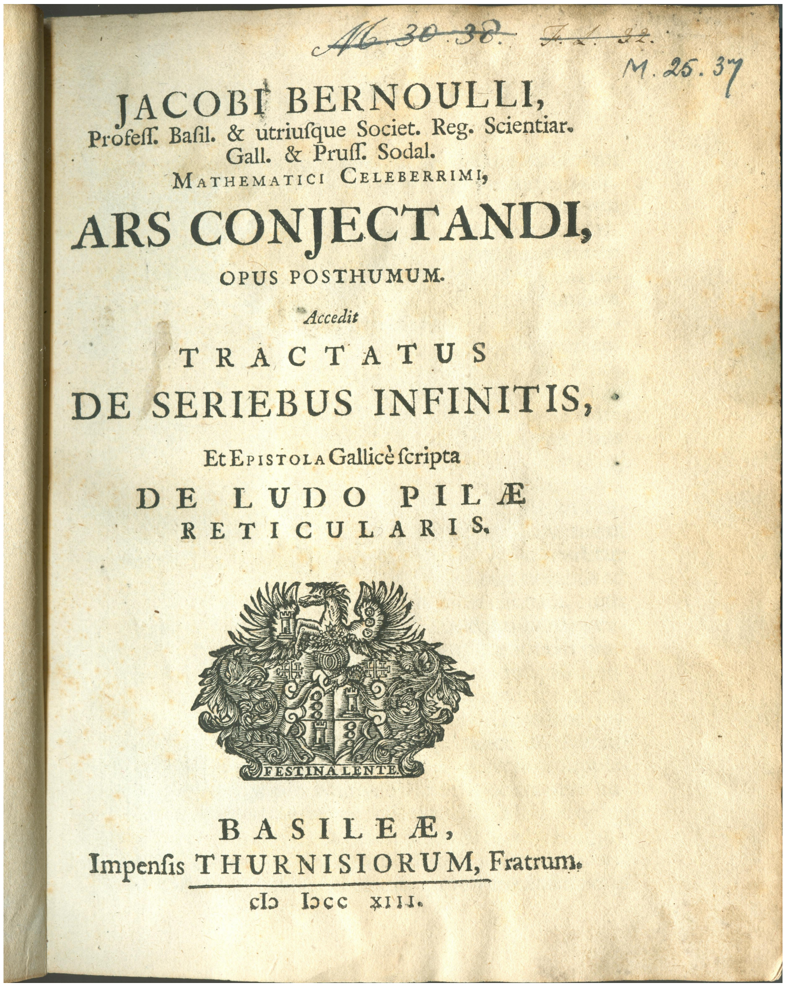
More people will have heard of its author. Jacob Bernouilli was one of a huge mathematical family of Bernoullis. In physics, aircraft engineers base everything they do on Bernoulli's principle. It explains how aircraft wings give lift, is the basis of fluid dynamics, and was discovered by Jacob's nephew Daniel Bernoulli.

Jacob Bernoulli (1654-1705)

Johann Bernoulli made important advances in mathematical calculus. He was Jacob's younger brother – the two fell out bitterly. Johann fell out with his fluid-dynamics son Daniel, too, and even falsified the date on a book of his own to try to show that he had discovered the principle first.

But our statistical Bernoulli is Jacob. In the higher reaches of pure mathematics he is loved for Bernoulli numbers, which are fiendishly complicated things which I do not pretend to understand but which apparently underpin number theory. In statistics, his contribution was two-fold: Bernoulli trials are, essentially, coinflips repeated lots of times. Toss a fair coin ten times, and you might well get 6 heads and four tails rather than an exact 5/5 split. Toss 100 times and you are quite unlikely to get 60 heads and 40 tails. The more times you toss the coin, the closer you will get to a 50-50 result.

His second statistical result was more fundamental, though related. Suppose you have an urn with 3000 white pebbles and 2000 black pebbles. You take out a pebble, look at its colour, and return it. Then do it again; and again; and again. After ten times you might guess that there were 2/3 as many black pebbles as white; after 1000 times you might feel a bit more sure of it. Can you do this so often that you become absolutely sure – morally certain, as Bernoulli put it - that the pebbles in the vase were actually in the ratio of 3 to 2? Or would that conclusion forever remain just a conjecture?



JACOBI BERNOULLI,
Profess. Basil. & utriusque Societ. Reg. Scientiar. Gall. & Pruss. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicè scripta
DE LUDO PILÆ RETICULARIS.

FESTINA LENTE

BASILEÆ,
Impensis THURNISIORUM, Fratrum.
cIↃ IↃCC XIII.

Ars Conjectandi, Title page.

Courtesy Gonville and Caius College, Cambridge.

If it is just a conjecture, then all of statistics is built on sand. Happily, Bernoulli showed it was more than a conjecture; he spent years thinking about it, managed to prove it was true – and when he had done so he called it his Golden Theorem as it was the crown of his life's work. The more time you repeat a series of experiments like this, the closer your result will get to the true one. Statisticians are rather pleased that he proved it. If it had stayed a conjecture, there would have been no need to believe anything (statistical) that a statistician told you.

We shall have a major scholarly piece on Ars Conjectandi in our June issue, out on paper and on this site shortly. A challenge: can you correct something that Jacob Bernoulli got wrong? It stayed wrong for nearly 300 years until our author, Professor Antony Edwards, spotted it and corrected it.

Here is the problem: It is a simple exercise in schoolboy probability. It is Problem XVII in Part III of Bernoulli's book. For those who would like to try their hand, the problem is as follows.

Puncta		Nummi		Casus communes.
4	32	120	180	1
5	31	100	32	16
6	30	30	25	52
7	29	24	24	128
8	28	18	16	245
9	27	10	12	416
10	26	6	8	664
11	25	6	6	976
12	24	6	4	1369
13	23	5	4	1776
14	22	3	3	2204
15	21	3	3	2560
16	20	3	3	2893
17	19	2	3	3088
18		2		3184

Bernoulli' table..

From Bernoulli's *Ars Conjectandi*

In a version of roulette, the wheel is surrounded by 32 equal pockets marked 1 to 8 four times over. Four balls are released and are flung at random into the pockets, no more than one in each. The sum of the numbers of the four occupied pockets determines the prize (in francs, say) according to a table which Bernoulli gives – it is on the right. The cost of a throw is 4 francs. What is the player's expectation? That is, how much, in the long run, can he expect to walk away with per game?

The left-hand columns in the table are the total four-ball score; centre columns are the paybacks for a four-franc stake; the right-hand columns are the number of combinations that could give rise to the score.

The answer Bernoulli gives in the book is 4 + 349/3596, which is 4.0971. Professor Edwards comes up with a different answer, which we shall give in his article in the magazine section of this site when the issue goes live in about a week. Which do you agree with?

And happy calculating…

Comments

Graham Wheeler

Assuming I've correctly amended Bernoulli's table, I find the answer to the problem is 4.006618.

Gonzalo Marí

I found the same value, 4.45800781

Dinesh Hariharan

Converges to 4.458, found in the most inelegant manner.

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luſoris B $\frac{4820}{9009}$, ſic ut ratio ſortium ſit, ut 4189 ad 4820. Unde perſpicuum fit, potiorem hujus quàm illius conditionem eſſe, ut maximè ſint qui ſecus exiſtiment, inque partes ipſius A tranſire malint.

PROBLEMA XVII.

Æſtimatio ſortis in alio quodam Aleæ genere.

Memini me olim tempore nundinarum quendam hic vidiſſe Circulatorem, qui ſequens aleæ genus in foro exponebat, eoque prætereuntes alliciebat. Diſcus erat orbicularis ad libellam compoſitus, versùs medium parumper acclivis; Limbum circumcingebant 32 loculi ſeu foraminula contigua & æqualia, quæ in quatuor diſtincta claſſes vel ſeries numeris ordine ab I uſque ad VIII quater adſcriptis ſignabantur; Medio diſci perpendiculariter imminebat fritillus. Fortunam periclitaturus per cavitatem fritilli quatuor demittebat globulos excipiendos in circumferentia diſci à totidem loculis, auferebatque præmium quod numeris horum loculorum in ſummam collectis dicatum conſpiciebat, majoris minoriſve pretii pro aggregati diverſitate, ut ex ſubjuncto laterculo apparet. Singuli autem globulorum jactus ipſi quaternis nummis redimendi erant. Quæritur ipſius expectatio?

Conſtat primò, quòd unoquovis globulorum jactu ad minimum 4, ad ſummum 32 puncta obtineri poſſunt, quorum utrumvis uno duntaxat caſu contingit, illud, ſi globuli ſinguli ſingulorum ordinum prima foramina ſubintrant, iſtud ſi ultima. Deinde obſervo, quòd caſus multiplicentur pro intermediis punctorum numeris, prout ab utrovis extremo 4 aut 32 magis recedunt, & quòd maximo caſuum numero ſit expoſitus numerus 18, medius arithmeticus inter 4 & 32; bini autem numeri à medio 18 ſupra infraque æqualiter remoti æquali quoque caſuum numero ſubſint. Tertiò conſidero, quòd foraminula, quæ quovis jactu globulos excipiunt, vel omnia quatuor ſignata eſſe poſſint eodem determinato numero; vel tria eodem, quar-

Puncta		*Nummi*		*Casus communes.*
4	32	120	180	1
5	31	100	32	16
6	30	30	25	52
7	29	24	24	128
8	28	18	16	245
9	27	10	12	416
10	26	6	8	664
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13	23	5	4	1776
14	22	3	3	2204
15	21	3	3	2560
16	20	3	3	2893
17	19	2	3	3088
18		2		3184

tum diverso: vel duo eodem, & reliqua duo alio eodem numero: vel duo eodem, & cætera duo diversis: vel denique omnia quatuor differentibus determinatis numeris; quorum quidem primum unico, alterum 16, tertium 36, quartum 96, ultimum 256 casibus accidere potest. Etenim cùm quaterni sint loculi homologi, sive eodem determinato numero putà I notati, si globulorum nonnulli putà tres ab istis loculis sunt excipiendi, liquet hoc tot casibus contingere posse, quot terniones in rebus 4 continentur, nempe quatuor; adeò ut si quartus insuper globulus in aliquem loculum alio numero, ex. gr. II. signatum se recipere debeat (quod ob quatuor uniones in rebus 4 rursum quatuor casibus evenit) concludi possit, quater quatuor seu 16 in universum casus existere, qui efficiant, ut tres globuli tres loculos n.° I signatos & simul quartus unum loculorum n.° II notatorum subintret. Quemadmodum etiam colligere promtum est, ob sex biniones rerum quatuor, sexies sex seu 36 casus haberi, quibus contingat, ut duo loculi n.° I conspicui à duobns, & duo n.° II notati ab aliis duobus globulis occupentur: nec non sexies quater quatuor, h. e. 96 casus, quibus duo loculi num. I à duobus, unus loculorum n.° II à tertio, & unus n.° III à 4to globulo occupetur: ac denique 4.4.4.4 h. e. 256 casus, quibus unus globulorum in loculum n.° I, alius in loculum n.° II, tertius in loculum III, & quartus in IV se recipiat. Ubi tandem notandum, quod ad variationes illas 24, quæ ex solâ 4 globulorum permutatione mutuâ oriuntur, non attendamus, quippe quæ insuper haberi possimt ceu totidem casus secundarii, ex quibus unusquisque primariorum conflatur.

His

His ita præmissis & intellectis inquirendum est in numerum casuum cuivis punctorum numero convenientem, eo ferè modo quo supra post Prop. 9. part. 1. ad numeros jactuum in tesseris investigandos usi fuimus; resolvendo vid. propositum punctorum numerum ob 4 globulos in 4 partes, quarum nulla octonarium superet (quòd loculis majores numeri non sint adscripti) idque omnibus modis possibilibus, ac deinde singulis modis juxta supra observata suos tribuendo casuum numeros; horum enim summa quæsitum exhibebit. At quoniam eâ ratione numerus casuum duntaxat pro dato punctorum numero inveniretur, nobis verò casuum notitia pro universis punctis necessaria est, poterimus aliam compendiosiorem inire viam, & omnia una operatione consequi, hoc modo:

In supremo sequentis Tabulæ margine scribantur ordine numeri punctorum à IV usque ad XVIII; sufficit enim horum determinasse casus, cùm singuli supra XVIII cum singulis infra in casuum multitudine, uti dictum, conveniant.

Ponamus, globulos omnes excipi 4 loculis homologis, erunt eorum numeri vel 4 unitates, vel 4 binarii, ternarii, quaternarii &c. quorum summæ sunt, 4, 8, 12, 16, &c. quare signetur in margine sinistro 1. 1. 1. 1 (cæteris 2. 2. 2. 2 &c. usq; ad 8. 8. 8. 8 mente subintellectis) & è regione sub singulis punctorum numeris IV. VIII. XII. XVI. &c. notentur singulæ unitates.

Ponamus tres globulos excipi loculis homologis, quartum diverso: erunt homologorum numeri vel tres unitates, vel totidem binarii, ternarii &c. Si tres unitates, quartus numerus erit vel binarius, vel ternarius, quaternarius &c. qui singuli juncti unitatibus summas efficiunt V. VI. VII. VIII. . . . XI; quocirca in margine signetur 1. 1. 1. 2 (reliquis 1. 1. 1. 3 &c. usque ad 1. 1. 1. 8 mente suppletis) & è regione sub punctis V. VI. VII. . . XI. scribatur 16. Si homologorum numeri sint tres binarii, quartus erit vel 1, vel 3, vel 4 &c. qui juncti binariis summas exhibent VII. IX. X. . . XIV; quare in margine ponatur 2. 2. 2. 1 (cæteris 2. 2. 2. 3 &c. subintellectis) & è regione sub singulis punctorum VII. IX. X. . . . XIV. rursus scribatur 16. Similiter etiam procedendum, ubi homologorum numeri sunt tres ternarii, existente quarto 1. 2. 4 aut 5 &c. aut tres quaternarii, existente quarto 1. 2. 3. aut 5 &c. aut tres quinarii &c. existente semper quarto uno reliquorum, scribendo nempe 16 sub singulis punctorum summis, quas additi 4 loculorum numeri efficiunt.

Ponamus porrò loculos globulorum duos homologos, & alios duos rursus homologos, sed à prioribus diversos: erunt numeri loculorum vel duæ unitates cum duobus binariis, ternariis, quaternariis &c. qui unitatibus juncti faciunt VI. VIII. X....XVIII: vel duo binarii cum 2 ternariis, quaternariis &c. qui additi binariis constituunt X. XII. XIV. &c. vel duo ternarii cum totidem quaternariis &c. vel duo quaternarii cum totidem quinariis &c. &c. idcirco notentur in margine Tabulæ 1. 1. 2. 2, 2. 2. 3. 3, 3. 3. 4. 4 &c. (cæteris 1. 1. 3. 3, 1. 1. 4. 4 &c. nec non 2. 2. 4. 4 &c. 3. 3. 5. 5 &c. compendii gratiâ omissis) è regione verò sub singulis numerorum tam expressorum quàm mente retentorum summis scribantur 36.

Pergamus deinde ponere loculos globulorum duos homologos, reliquos ab his & inter se diversos: erunt homologorum numeri rursus vel duæ unitates, vel duo binarii, ternarii &c. & si unitates, tertius erit vel binarius cum quarto ternario, quaternario, quinario &c. vel ternarius cum quarto quaternario, quinario &c. & ita consequenter: si duo binarii, tertius esse potest vel 1 cum quarto 3, 4, 5, 6 &c. vel 3 cum 4^{to} 4, 5, 6 &c. vel 4 cum 4^{to} 5, 6 &c. &c. si illi sunt duo ternarii, tertius existet vel 1 cum 4^{to} 2, 4, 5, 6 &c. vel 2 cum 4^{to} 4, 5, 6 &c. vel 4 cum $4.^{o}$ 5, 6 &c. &c. si illi sunt quaternarii, 3^{tius} poterit esse vel 1 cum 4^{to} 2, 3, 5 &c. vel 2 cum 4^{to} 3, 5 &c. &c. & ita pariter in reliquis omnibus, quamobrem primis harum combinationum 1. 1. 2. 3, 1. 1. 3. 4, &c. nec non 2. 2. 1. 3 &c. 3. 3. 1. 2 &c. in margine notatis & cæteris mente suppletis, scribantur sub singulis punctorum summis, quas singuli numerorum quaternarii efficiunt, 96.

Tandem etiam ponamus, loculos globulorum omnes differentibus numeris affectos esse; erunt ipsorum combinationes tales: 1. 2. 3 cum 4^{to} 4, 5, 6 &c. 1. 2. 4 cum 4^{to} 5, 6, 7 &c. &c. item 1. 3. 4:

1. 3. 5:

Puncta:

Ad pag. 172.

Combinat.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.
1.1.1.1.	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.
1.1.1.2	.	16	16	16	16	16	16	16							
2.2.2.1	.	.	.	16	.	16	16	16	16	16	16				
3.3.3.1	.	.	.	.	.	.	16	16	.	16	16	16	16	16	.
4.4.4.1	.	.	.	.	.	.	.	.	.	16	16	16	.	16	16
5.5.5.1	.	.	.	.	.	.	.	.	.	.	.	.	16	16	16
1.1.2.2	.	.	36	.	36	.	36	.	36	.	36	.	36	.	36
2.2.3.3	.	.	.	.	.	.	36	.	36	.	36	.	36	.	36
3.3.4.4	.	.	.	.	.	.	.	.	.	.	36	.	36	.	36
4.4.5.5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	36
1.1.2.3	.	.	.	96	96	96	96	96	96	.	.	.	.	.	.
1.1.3.4	.	.	.	.	.	96	96	96	96	96	.	.	.	.	.
1.1.4.5	.	.	.	.	.	.	.	96	96	96	96	.	.	.	.
1.1.5.6	.	.	.	.	.	.	.	.	.	96	96	96	.	.	.
1.1.6.7	.	.	.	.	.	.	.	.	.	.	.	96	96	.	.
1.1.7.8	.	.	.	.	.	.	.	.	.	.	.	.	.	96	.
2.2.1.3	.	.	.	.	96	96	96	96	96	96	.	.	.	.	.
2.2.3.4	.	.	.	.	.	.	.	96	96	96	96	96	.	.	.
2.2.4.5	.	.	.	.	.	.	.	.	.	96	96	96	96	.	.
2.2.5.6	.	.	.	.	.	.	.	.	.	.	.	96	96	96	.
2.2.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
3.3.1.2	.	.	.	.	.	96	.	96	96	96	96	96	.	.	.
3.3.2.4	.	.	.	.	.	.	.	.	96	96	96	96	96	.	.
3.3.4.5	.	.	.	.	.	.	.	.	.	.	.	96	96	96	96
3.3.5.6	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
4.4.1.2	.	.	.	.	.	.	.	96	96	.	96	96	96	96	.
4.4.2.3	.	.	.	.	.	.	.	.	.	96	.	96	96	96	96
4.4.3.5	.	.	.	.	.	.	.	.	.	.	.	.	96	96	96
5.5.1.2	.	.	.	.	.	.	.	.	.	96	96	96	.	96	96
5.5.2.3	.	.	.	.	.	.	.	.	.	.	.	96	96	.	96
5.5.3.4	.	.	.	.	.	.	.	.	.	.	.	.	.	96	.
6.6.1.2	.	.	.	.	.	.	.	.	.	.	.	96	96	96	96
6.6.2.3	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
7.7.1.2	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
1.2.3.4	.	.	.	.	.	.	256	256	256	256	256	.	.	.	.
1.2.4.5	.	.	.	.	.	.	.	.	256	256	256	256	.	.	.
1.2.5.6	.	.	.	.	.	.	.	.	.	.	256	256	256	.	.
1.2.6.7	.	.	.	.	.	.	.	.	.	.	.	.	256	256	.
1.2.7.8	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
1.3.4.5	.	.	.	.	.	.	.	.	.	256	256	256	256	.	.
1.3.5.6	.	.	.	.	.	.	.	.	.	.	.	256	256	256	.
1.3.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	256	256
1.4.5.6	.	.	.	.	.	.	.	.	.	.	.	.	256	256	256
1.4.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
2.3.4.5	.	.	.	.	.	.	.	.	.	.	256	256	256	256	.
2.3.5.6	.	.	.	.	.	.	.	.	.	.	.	.	256	256	256
2.3.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
2.4.5.6	.	.	.	.	.	.	.	.	.	.	.	.	.	256	256
3.4.5.6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
Summa Casuum	1	16	52	128	245	416	664	976	1369	1776	2204	2560	2893	3088	3184

Summa omnium Casuum — — 35960.

1.3.5: &c. 1.4.5 &c. cum 4^{to} &c. nec non 2.3.4: 2.3.5 &c. cum quarto differenti numero &c. &c. &c. usque ad 3.4.5.6 quâ cum omnes possibiles combinationes sunt completæ: quocirca primis harum combinationum in margine expressis & prætermissis reliquis, notentur è regione sub singulis quaternorum numerorum summis, 256; prout hæc omnia in adjunctâ Tabulâ præstita cernuntur.

Additis igitur in unam summam, qui in eâdem serie perpendiculari sibi respondent, numeris, habebuntur omnes punctorum in vertice scriptorum casus, videl. 1 casus pro punctis IV, 16 casus pro punctis V, 52 pro punctis VI; & ita deinceps usque ad puncta XVIII, qui numerus bis 16, quater 36, decies 96 & octies 256, id est, in universum 3184 casibus expositus est. Et quoniam numeri punctorum supra XVIII cum reliquis infra, singuli cum singulis, putà XIX cum XVII, XX cum XVI &c. in numero casuum conveniunt, ut initio monuimus & ostensu facile est, sequitur, si collecti à punctis IV ad XVII casuum numeri duplentur, duploque 32776 addantur 3184 casus punctorum XVIII, aggregatum 35960 exhibiturum summam omnium omnino casuum. Quòd autem enumeratio ritè facta sit, nullaque combinationum prætermissa, vel inde patet, quòd numerus quaternionum in rebus (putà loculis) 32, præcisè idem reperitur; est enim ille, per Cap. 4. part. 2,

$$\frac{32.31.30.29}{1.2.3.4} \infty 35960.$$

Inventis sic numeris casuum pro quovis punctorum numero, cætera oppidò levia sunt, & expediuntur per Prop. 3. part. 1. multiplicando videl. singulos casuum numeros per singula præmia, quæ istis casibus acquiruntur: nempe (cùm punctis IV in sup. laterculo tribuantur nummi 120, punctis XXXII nummi 180, punctis V nummi 100, punctis XXXI nummi 32, punctis VI 30, punctis XXX 2 &c.) multiplicando 1 casum per 120, rursusque per 180; 16 casus per 100, iterumque per 32; 52 casus per 30, nec non per 25 &c. sive brevius, 1 per 300 ∞ 120 + 180, 16 per 132 ∞ 100 + 32, 52 per 55 ∞ 30 + 25, atque ita deinceps usque ad 3184 per 2: ac tandem dividendo omnium productorum summam per summam omnium casuum 35960. Sic enim exibunt in quotiente pro expectatione aleatoris nummi 4 $\frac{345}{556}$: unde cùm ipse ex hypoth. solis 4 nummis jactum redimerit, apparet potiorem illius quàm circulatoris sortem esse, istumque proin hoc aleæ genere, ni præmia minuat, non multum lucrari posse.

PROBLEMA XVIII.

De Ludo chartarum, vulgò Trijaques.

Usitatissimum est inter Germanos ludi genus, quod *Trijaques* appellatur, & affinitatem quandam habet cum Gallorum *Brelan*: Sumuntur ex Ludo chartarum folia 24 (rejectis cæteris) ex unaquaque scil. specie sex; nimirum *Novenarii*, *Denarii*, *Famuli*, *Heræ*, *Reges* & *Monades*, quæ suis posthac literis initialibus N. D. F. H. R. M denotabuntur, & hunc dignitatis ordinem inter se servant: Primas tenet *Monas*, sequitur *Rex*, inde *Hera*, *Famulus*, *Denarius*; sed omnibus supereminent *Novenarii* unà cum *Famulo trifolii* (quem proin etiam Novenariis accensemus, sic ut 5 habeamus Novenarios, at 3 tantùm Famulos). Novenariorum præstantia, similis ferè horum, quos in ludo Hispanico *Jeu de l'Hombre* dicto *Matadors*, latrones, homicidas sive sicarios appellant, in eo consistit, ut cujusvis dignitatis & speciei chartis accenseantur: sic duo Novenarii cum Monade, aut unus cum duabus juncti tres Monadas, seu *Trigam* (*un Tricon*) Monadum efficiunt: unus, duo vel tres Novenarii stipati tribus, duobus unove Regibus *Quadrigam* Regum constituunt: unus duove Novenarii in consortio trium duorumve ejusdem speciei foliorum, quaterna illius speciei exhibent, ex. gr. quaterna corda, spicula, trifolia &c. cujusmodi chartarum complexio *Fluvius*, ein Fluß / dici consuevit, qui præterea numero punctorum æstimatur; numerantur autem pro Novenario aut Monade puncta 11, pro cæterarum dignitatum chartis singulis puncta 10. Modus ludendi talis:

Singulis colludentium ordine bina distribuuntur folia, quibus clam inspectis liberum est primo arbitrariam pecuniæ summam deponere, quocum si congredi velit alter, tantundem deponet, aut metical statements by iterating certain reflection principles. Franzén's other book, *Inexhaustibility: A Non-exhaustive Treatment* (ASL Lecture Notes in Logic #16, 2004) contains an excellent exposition of the incompleteness theorems, and the reader is led step-by-step through the technical details needed to establish a significant part of Feferman's completeness results for iterated reflection principles for ordinal logics.

Torkel Franzén's untimely death on April 19, 2006 came shortly before he was to attend, as an invited lecturer, the Gödel Centenary Conference, "Horizons of Truth," held at the University of Vienna later that month. This, and his invitations to speak at other conferences featuring a tribute to Gödel, testifies to the growing international recognition that he deserved for these works.

ACKNOWLEDGMENTS

I thank Solomon Feferman for substantive and insightful correspondence during the preparation of this review, and Robert Crease, Patrick Grim, Robert Shrock, Lorenzo Simpson, and Theresa Spörk-Greenwood for their intellectual and material support for my participation in the Gödel Centenary "Horizons of Truth" Conference in Vienna.

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The Art of Conjecturing together with *Letter to a Friend on Sets in Court Tennis*

by Jacob Bernoulli

translated with an introduction and notes by Edith Dudley Sylla

BALTIMORE, THE JOHNS HOPKINS UNIVERSITY PRESS, 2006, xx + 430 PP. £46.50 ISBN 0-8018-8235-4.

REVIEWED BY A. W. F. EDWARDS

In 1915 the young statistician R. A. Fisher, then 25, and his former student friend C. S. Stock wrote an article [1] bewailing the contemporary neglect of *The Origin of Species*:

> So melancholy a neglect of Darwin's work suggests reflections upon the use of those rare and precious possessions of man—great books. It was, we believe, the custom of the late Professor Freeman [2] to warn his students that mastery of one great book was worth any amount of knowledge of many lesser ones. The tendency of modern scientific teaching is to neglect the great books, to lay far too much stress upon relatively unimportant modern work, and to present masses of detail of doubtful truth and questionable weight in such a way as to obscure principles. . . . How many biological students of today have read *The Origin*? The majority know it only from extracts, a singularly ineffective means, for a work of genius does not easily lend itself to the scissors; its unity is too marked. Nothing can really take the place of a first-hand study of the work itself.

With her translation of Jacob Bernoulli's *Ars Conjectandi* in its entirety Edith Sylla now makes available to English-speakers without benefit of Latin another great book hitherto known mostly from extracts. As she rightly observes, only thus can we at last see the full context of *Bernoulli's theorem*, the famous and fundamental limit theorem in Part IV that confirms our intuition that the proportions of successes and failures in a stable sequence of trials really do converge to their postulated probabilities in a strict mathematical sense, and therefore may be used to estimate those probabilities.

However, I must resist the temptation to review *Ars Conjectandi* itself and stick to Sylla's contribution. She thinks that it *'deserves to be considered the founding document of mathematical probability'*, but I am not so sure. That honour belongs to Bernoulli's predecessors Pascal and Huygens, who *mathematized* expectation half a century earlier; Bernoulli's own main contribution was 'The Use and Application of the Preceding Doctrine in Civil, Moral, and Economic Matters' (the title of Part IV) and the associated theorem. It would be more true to say that *Ars Conjectandi* is the founding document of mathematical *statistics*, for if Bernoulli's theorem were not true, that enterprise would be a house of cards. (The title of a recent book by Andres Hald says it all: *A History of Parametric Statistical Inference from Bernoulli to Fisher, 1713–1935* [3].)

When I first became interested in Bernoulli's book I was very fortunately placed. There was an original edition in the college library (Gonville and Caius College, Cambridge) and amongst the other Fellows of the college was Professor Charles Brink, the University's Kennedy Professor of Latin. Though I have school Latin I was soon out of my depth, and so I consulted Professor Brink about passages that particularly interested me. Charles would fill his pipe, settle into his deep wing-chair and read silently for a while. Then, as like as not, his opening remark would be 'Ah, yes, I remember Fisher asking me about this passage'. Fisher too had been a Fellow of Caius.

Now, at last, future generations can set aside the partial, and often amateur, translations of *Ars Conjectandi* and enjoy the whole of the great work professionally translated, annotated, and introduced by Edith Sylla, in a magisterial edition beautifully produced and presented. She has left no stone unturned, no correspondence unread, no secondary literature unexamined. The result is a work of true scholarship that will leave every serious reader weak with admiration. Nothing said in criticism in this review should be construed as negating that.

The translation itself occupies just half of the long book, 213 pages. Another 146 pages are devoted to a preface and introduction, and 22 to a 'translator's commentary'. Next come 41 pages with a translation of Bernoulli's French *Letter to a Friend on Sets in Court Tennis* which was published with *Ars Conjectandi* and which contains much that is relevant to the main work; a translator's commentary is again appended. Finally, there is a full bibliography and an index.

In her preface Sylla sets the scene and includes a good survey of the secondary literature (Ivo Schneider's chapter on *Ars Conjectandi* in *Landmark Writings in Western Mathematics 1640–1940* [4] appeared just too late for inclusion). Her introduction 'has four main sections. In the first, I review briefly some of the main facts of Jacob Bernoulli's life and its social context. . . . In the second, I discuss

Bernoulli's other writings insofar as they are relevant. . . . In the third, I describe the conceptual backgrounds. . . . Finally, in the fourth, I explain the policies I followed in translating the work.' The first and second parts are extremely detailed scholarly accounts which will be standard sources for many years to come. The third, despite its title 'Historical and Conceptual Background to Bernoulli's Approaches in *Ars Conjectandi*', turns into quite an extensive commentary in its own right. Its strength is indeed in the discussion of the background, and in particular the placing of the famous 'problem of points' in the context of early business mathematics, but as commentary it is uneven.

Perhaps as a consequence of the fact that the book has taken many years to perfect, the distribution of material between preface, introduction, and translator's commentary is sometimes hard to understand, with some repetition. Thus one might have expected comments on the technical problems of translation to be included under 'translator's commentary', but most—not all—of it is to be found in the introduction. The distribution of commentary between these two parts is confusing, but even taking them together there are many lacunae.

The reason for this is related to Sylla's remark at the end of the introduction that 'Anders Hald, A. W. F. Edwards, and others, in their analyses of *Ars Conjectandi*, consistently rewrite what is at issue in modern notation. . . . I have not used any of this modern notation because I believe it obscures Bernoulli's actual line of thought.' I and others have simply been more interested in Bernoulli's mathematical innovations than in the historical milieu, whose elucidation is in any case best left to those, like Sylla, better qualified to undertake it. Just as she provides a wealth of information about the latter, she often passes quickly over the former.

Thus (pp. 73, 345) she has no detailed comment on Bernoulli's table (pp. 152–153) enumerating the frequencies with which the different totals occur on throwing n dice, yet this is a brilliant tabular algorithm for convoluting a discrete distribution, applicable to any such distribution. In 1865 Todhunter [5] 'especially remark[ed]' of this table that it was equivalent to finding the coefficient of x^m in the development of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^n$, where n is the number of dice and m the total in question. Again (pp. 74–75, 345), she has nothing to say about Bernoulli's derivation of the binomial distribution (pp. 165–167), which statisticians rightly hail as its original appearance. Of course, she might argue that as Bernoulli's expressions refer to expectations it is technically not a *probability* distribution, but that would be to split hairs. Statisticians rightly refer to 'Bernoulli trials' as generating it, and might have expected a reference.

Turning to Huygens's Vth problem (pp. 76, 345), she does not mention that it is the now-famous 'Gambler's ruin' problem posed by Pascal to Fermat, nor that Bernoulli seems to be floundering in his attempt at a general solution (p. 192). And she barely comments (p. 80) on Bernoulli's polynomials for the sums of the powers of the integers, although I and others have found great interest in them and their earlier derivation by Faulhaber in 1631, including the 'Bernoulli numbers'. Indeed, it was the mention of Faulhaber in *Ars Conjectandi* that led me to the discovery of this fact (see my *Pascal's Arithmetical Triangle* and references therein [6]; Sylla does give some relevant references in the translator's commentary, p. 347).

I make these remarks not so much in criticism as to emphasize that *Ars Conjectandi* merits deep study from more than one point of view.

Sylla is probably the only person to have read Part III right through since Isaac Todhunter and the translator of the German edition in the nineteenth century. One wonders how many of the solutions to its XXIV problems contain errors, arithmetical or otherwise. On p. 265 Sylla corrects a number wrongly transcribed, but the error does not affect the result. Though one should not make too much of a sample of one, my eye lit upon Problem XVII (pp. 275–81), a sort of roulette with four balls and 32 pockets, four each for the numbers 1 to 8. Reading Sylla's commentary (p. 83) I saw that symmetry made finding the expectation trivial, for she says that the prize is 'equal to the sum of the numbers on the compartments into which [the] four balls fall' (multiple occupancy is evidently excluded). Yet Bernoulli's calculations cover four of his pages and an extensive pull-out table.

It took me some time to realize that Sylla's description is incorrect, for the sum of the numbers is not the prize itself, but an indicator of the prize, according to a table in which the prizes corresponding to the sums are given in two columns headed *nummi*. Sylla reasonably translates this as 'coins', though 'prize money' is the intended meaning. This misunderstanding surmounted, and with the aid of a calculator, I ploughed through Bernoulli's arithmetic only to disagree with his answer. He finds the expectation to be 4 349/3596 but I find 4 153/17980 (4.0971 and 4.0085). Bernoulli remarks that since the cost of each throw is set at 4 'the player's lot is greater than that of the peddler' but according to my calculation, only by one part in about 500. I should be glad to hear from any reader who disagrees with my result. Sylla has translated *Circulator* as 'peddler' ('pedlar' in British spelling) but 'traveler' might better convey the sense, especially as Bernoulli uses the capital '*C*'.

And so we are led to the question of the translation itself. How good is it? I cannot tell in general, though I have some specific comments. The quality of the English is, however, excellent, and there is ample evidence of the care and scholarly attention to detail with which the translation has been made. I may remark on one or two passages.

First, one translated in my *Pascal's Arithmetical Triangle* thus:

> This Table [Pascal's Triangle] has truly exceptional and admirable properties; for besides concealing within itself the mysteries of Combinations, as we have seen, it is known by those expert in the higher parts of Mathematics also to hold the foremost secrets of the whole of the rest of the subject.

Sylla has (p. 206):

> This Table has clearly admirable and extraordinary properties, for beyond what I have already shown of the mystery of combinations hiding within it, it is known to those skilled in the more hidden parts of geometry that the most important secrets of all the rest of mathematics lie concealed within it.

Latin scholars will have to consult the original to make a judgment, but, settling down with a grammar and a dictionary 25 years after my original translation (with which Professor Brink will have helped), I think mine better and closer to the Latin. I might now change 'truly' to 'wholly' and prefer 'mystery' in the singular (like Sylla), as in the Latin, as well as simply 'higher mathematics'. But her 'geometry' for *Geometria* is surely misleading, for in both eighteenth-century Latin and French the word encompassed the whole of mathematics.

Second, there is an ambiguity in Sylla's translation (p. 194) of Bernoulli's claim to originality in connection with a 'property of figurate numbers'. Is he claiming the property or only the demonstration? The latter, according to note 20 of chapter 10 of *Pascal's Arithmetical Triangle.*

Third, consider Sylla's translation (p. 329) of Bernoulli's comment on his great theorem in part IV:

> This, therefore, is the problem that I have proposed to publish in this place, after I have already concealed it for twenty years. Both its novelty and its great utility combined with its equally great difficulty can add to the weight and value of all the other chapters of this theory.

Did Bernoulli actively *conceal* it? In colloquial English I think he just *sat on it* for twenty years (*'pressi'*); De Moivre [7] writes 'kept it by me'. And does it add weight and value, or add *to* the weight and value? De Moivre thought the former (actually 'high value and dignity'). This is also one of the passages on which I consulted Professor Brink. His rendering was:

> This then is the theorem which I have decided to publish here, after considering it for twenty years. Both its novelty and its great usefulness in connexion with any similar difficulty can add weight and value to all other branches of the subject.

In one instance Sylla unwittingly provides two translations of the same Latin, this time Leibniz's (p. 48n and p. 92). The one has 'likelihood' and the other 'verisimilitude' for *'verisimilitudo'*. And just one point from the French of the 'Letter to a Friend' (p. 364): surely 'that it will finally be as probable as *any* given probability', not *'all'*.

Finally, in view of the fact that this irreplaceable book is sure to remain the standard translation and commentary for many years to come, it may be helpful to note the very few misprints that have come to light: p. xvi, lines 1 and 2, De Moivre has lost his space; p. 73, line 14, Huygens has lost his 'g'; p. 152, the table headings are awkwardly placed and do not reflect the original in which they clearly label the initial columns of Roman numerals; p. 297*n*, omit *diario*; and in the Bibliography, p. 408, the reference in Italian should have *'Accademia'*, and Bayes's paper was published in 1764; p. 415, Kendall not Kendell; and, as a Parthian shot from this admiring reviewer, on p. 414 the title of my book *Pascal's Arithmetical Triangle* should not be made to suffer the Americanism *'Arithmetic'*.

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James Joseph Sylvester: Jewish Mathematician in a Victorian World

by Karen Hunger Parshall

THE JOHNS HOPKINS UNIVERSITY PRESS, BALTIMORE. 2006. xiii + 461 PP. $69.95, ISBN: 0-8018-8291-5.

REVIEWED BY TONY CRILLY

James Joseph Sylvester (1814–1897) is well known to mathematicians. Was he not the scatter-brained eccentric who wrote a poem of four hundred lines, each rhyming with Rosalind? And, lecturing on it, spent the hour navigating through his extensive collection of footnotes, leaving little time for the poem itself? Another story told by E. T. Bell is of Sylvester's poem of regret titled "A missing member of a family of terms in an algebraical formula." Such scraps inevitably evoke a smile today, but is his oddity all there is—stories and tales to spice a mathematical life? Bell's essays in *Men of Mathematics* have been influential for generations of mathematicians, but his snapshots could not claim to be rounded biographies in any sense. This, then, is a review of the first full-length biography of the extraordinary mathematician J. J. Sylvester.

How can we judge a mathematical biography? On the face of it, writing the life of a mathematician is straightforward: birth, mathematics, death. Thus flows the writing formula: describe the mathematics, and top and tail with the brief biographical facts and stories. A possible variant is the briefly written life, followed by the mathematical heritage. There are many approaches, but these are consistent with William Faulkner's estimate of literary biography, "he wrote the novels and he died." According to this hard-line view, biography should not even exist. Yet Sylvester deserves to be rescued from Bell's thumbnail sketch of the "Invariant Twins" in which he lumped Cayley and Sylvester together in the same chapter.

Perhaps only in the genre of mathematical biography do possible subjects outnumber potential authors. A stimulating article on writing the life of a mathematician, and an invitation to contribute, has recently been published by John W. Dawson, the biographer of Kurt Gödel.[1] Writing about another person's life is a voyage of discovery about one's own life, and surely the biographer is different at the end of such a project. Writing about a period of history different from one's own also involves some exotic time travelling.

A central problem for writers whose subjects' lives were bounded by technical material is to integrate technical developments with the stories of those lives. This is almost obvious, but it is

Ars conjectandi three hundred years on

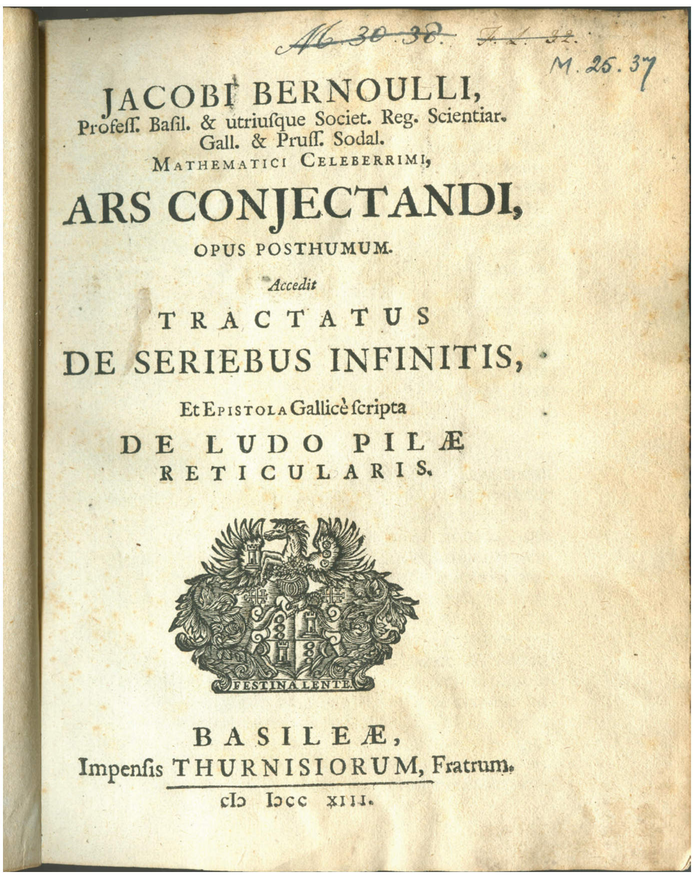
This year sees the 300th anniversary of Bernoulli's *Ars conjectandi – The Art of Conjecturing*. It transformed gamblers' "expectations" into modern mathematical probabilities. More importantly, it sets forth what Bernoulli called his "golden theorem" – the law of large numbers – which underpins the whole of statistical inference. **Professor Anthony Edwards** digs deep.

Bernoulli's *Ars conjectandi* appeared posthumously in 1713, eight years after its author's death. It was written in Latin; it was published in Basle, in Switzerland, Bernoulli's birthplace and the town where he spent most of his life as a professor of mathematics. Its four sections introduce, among other things, the modern concept of probability and Bernoulli's weak law of large numbers, the first limit theorem in probability, which initiated discussion of how one could draw reliable inferences from statistical data. And it is the founding document of mathematical statistics.

When I first became interested in Bernoulli's book I was very fortunately placed. There was an original edition in the college library (Gonville & Caius College, Cambridge). It was quite small – about 6″ × 8″ – and not distinguished to look at (I do not think the edition is particularly rare), but it had a nice leather binding and was reasonably worn through use. Amongst the other Fellows of the college was Professor Charles Brink, the University's Kennedy Professor of Latin. Though I have school Latin I was soon out of my depth, and used to consult Professor Brink about passages that particularly interested me. Charles would fill his pipe, settle into his deep wing-chair and read silently for a while. Then, as like as not, his opening remark would be "Ah, yes, I remember Fisher asking me about this passage". R.A. Fisher too had been a Fellow of Caius.

So what is this book, and why is it so fundamental to the history and to the development of statistics? Before we examine it, we should establish which Bernoulli wrote it. The Bernoulli family in Basle was a large one. Not only did it contain many mathematicians (at least ten of note are listed in one scholarly article[1]) but they often shared Christian names too. There was the further question of which language was appropriate for the Christian name of a family equally at home in Latin, German and French. The author of *Ars conjectandi* was Jacobus (1655–1705), which is the name on his tombstone. He was Jakob in German and Jacques in French. Abraham De Moivre (in the first edition of *The Doctrine of Chances*, 1718) had no hesitation in calling him James in English. All this was consistent with the accepted renderings of biblical names in translations of the Bible, in this case of Jesus's two disciples called James. Edith Sylla, in the first complete translation of *Ars conjectandi* into English[2], used "Jacob" on the grounds that "this is how the name appears on the title page". But what appears there is actually "Jacobi", this being the genitive of Jacobus appropriate to the grammatical context. As to the surname, De Moivre in fact wrote "James Bernoully", and this spelling of it occurs in letters. "Bernoullj" appears sometimes as well, and "Bernouilli" was quite common at one time, perhaps from French influence, but certainly known in England.

The Bernoulli family was apparently quarrelsome as well as large. After finishing his Master of Arts degree in 1671, "our" Bernoulli – let us



JACOBI BERNOULLI,
Profeſſ. Baſil. & utriuſque Societ. Reg. Scientiar.
Gall. & Pruſſ. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicè ſcripta
DE LUDO PILÆ
RETICULARIS.

BASILEÆ,
Impenſis THURNISIORUM, Fratrum.
cIↄ Iↄcc xiii.

Title page of Bernoulli's *Ars conjectandi*, the copy in Gonville & Caius College, Cambridge. Courtesy of Gonville & Caius call him James – studied theology until 1676. At the same time he was studying mathematics and astronomy – but doing so in secret, against the will of his father, who was a drug merchant. (James was the first of the Bernoulli clan to break into mathematics.) His younger brother John (or Johannes, or Johann) was also an accomplished mathematician, but the two quarrelled bitterly and notoriously. When James died suddenly in 1705 the book was unfinished. John was perhaps the most competent person to have completed the book, but the effects of the quarrel lasted beyond the grave and prevented John from getting access to the manuscript. The publishers hoped that James' son Nicholas might complete it. (He also has two nephews called Nicholas, which confused scholars hugely. It was long thought that it was one of the nephews who saw the book through the printers; it is only recently that Sylla has established that it was the son. I told you the names were confusing.) But Nicholas advised them to publish the work as it stood. In a preface he invited Pierre Rémond de Montmort and Abraham De Moivre, already know for their books *Essay d'analyse sur les jeux de hazard* (1708), and *De mensura sortis* (1712), respectively, to take up the challenge of extending James's calculus of probability to "economics and politics" as he had intended.

Since Sylla's translation in 2006 it has been possible to read the entire book in English for the first time. It is a book in four parts. The famous limit theorem, perhaps the mathematical justification for almost all of statistics, comes in Part IV; but the whole of *Ars conjectandi* needs celebrating at its tercentenary. Parts I, II and III essentially constitute a textbook on the emerging mathematics of combinations and probability. Had they been published soon after writing they might have had greater impact, but because of the post-mortem delay in publication the works of Montmort and De Moivre somewhat pre-empted them. Part I, for example, contains the binomial distribution for general chances which is named after Bernoulli (as we shall call James from now on) and which is often attributed to him, and this may indeed be just, since he probably found it between 1685 and 1689. Yet its actual first publication was by De Moivre in *De mensura sortis*, followed by Montmort in his second edition of 1713, the year *Ars conjectandi* finally appeared.

Part I of Bernoulli's book is a mainly a commentary on a book by Christiaan Huygens, who is known to schoolchildren as the inventor of the pendulum clock but to historians of science and mathematics as considerably more. Huygens's *De ratiociniis in ludo aleae* (*On Reasoning in Games of Dice*) came out in 1657. Part I of Bernoulli's book is entitled "Annotations on Huygens's Treatise"; it is 71 pages long in the original and it reprints Huygens's work with added commentaries of his own. First Bernoulli gives Huygens's Propositions I–IX concerning the problem of points, with his own annotations. The problem of points is, essentially, how to divide up the stakes fairly if a game of chance has to be abandoned halfway through – if a player has to leave for some reason – and had been debated since the late middle ages.

After Proposition VII, Bernoulli has added a table for the division of stakes between two players (he derives the table in Part II), whilst the table for three players after Proposition IX is Huygens's own. Propositions X–XIV consider dice throws, and after his annotation on Proposition XII Bernoulli devotes a section to developing the binomial distribution for general chances. He describes what are now known as "Bernoulli trials" – essentially the fundamentals of coin-tossing, dice-throwing and similar gambles (see box below). Huygens ended his book with five problems for solution, of which we may note the fifth in particular because it is a problem set by Pascal for Fermat (though neither Huygens nor Bernoulli mention this) which became famous as the "gambler's ruin", the first problem involving the duration of play (see box). Huygens gave the solution without any explanation, and Bernoulli, after several pages of discussion, arrives at a general solution but "I leave the demonstration of this result to the resolution of the reader". Thus ends Part I.

Part II, entitled "Permutations and Combinations", is 66 pages long. It gives the usual rules for the number of ways in which n objects – coloured balls and the like – can be put in order ($n!$), and for the case where a, b, c, … of the objects are alike ($n!/(a!b!c!\ldots)$). Chapter II is on the combinatorial rules "$2^n - 1$" and "$2^n - n - 1$", while Chapter III is on combinations of different things taken 1, 2, 3 or more at a time and on the figurate numbers "by which these matters may be treated" – "figurate" numbers being those that

Bernoulli trials and gambler's ruin

A Bernoulli trial is an experiment which can have one of only two outcomes. A tossed coin can come down either heads or tails; a penalty shot at goal can either score or fail to score; a child can be either a girl or a boy. The outcomes can be called success or failure; and in a series of repeated Bernoulli trials the probability of success and failure remain constant. A Bernoulli process is one that repeatedly performs independent but identical Bernoulli trials, for instance by tossing a coin many times. An obvious use of it is to check whether a coin is fair.

Gambler's ruin is an idea first addressed by Pascal, who put it to Fermat before Huygens put it into his book. One way of stating it is as follows. If you play any gambling game long enough, you will eventually go bankrupt. And this is true even if the odds in the game are better than 50–50 for you – as long as your opponent has unlimited resources at the bank.

Imagine a gamble where you and your opponent spin a coin; and the loser pays the winner £1. The game continues until either you or your opponent has all the money. Suppose you start with a bankroll of £10, and your opponent has a bankroll of £20. What are the probabilities that (a) you, and (b) your opponent, will end up with all the money?

This is the question that Huygens and Bernoulli addressed. The answer is that the player who starts with more money has more chance of ending up with all of it. The formula is:

$$P_1 = n_1/(n_1 + n_2)$$

and

$$P_2 = n_2/(n_1 + n_2)$$

where n_1 is the amount of money that player 1 starts with, and n_2 is the amount that player 2 starts with, and P_1 and P_2 are the probabilities that player 1 or player 2 your opponent wins.

In this case, you, starting with £10 of the £30 total, stand 10/(10+20) = 10/30 = 1 chance in 3 of walking away with the whole £30; and your opponent stands twice that chance of doing so. Two times out of three he will bankrupt you.

But if you do happen to be the one who walks away with the £30, and if you play the game again, and again, and again, against different opponents or the same one who has borrowed more money – eventually you will lose the lot.

It follows that if your own capital is finite (as, sadly, it will be) and if you are playing against a casino with vastly more capital than you, if you carry on playing for long enough you are virtually certain to lose all your money. (The casino can additionally impose other limits, on such things as the size of bets, just to make the result even more general and even more certain.)

Perhaps surprisingly, this is true even if the odds in the game are stacked in your favour. Eventually there will be a long enough unfavourable run of dice, coins or the roulette wheel to bankrupt you. Infinite capital will overcome any finite odds against it.

can be arranged to make triangles, tetrahedra, and their higher-dimensional equivalents. The demonstrations are not as successful as Pascal's in his *Traité du triangle arithmétique* of 1665, of which Bernoulli was unaware. In a *Scholium* to Chapter III Bernoulli diverts into a discussion of the formulae for the sums of the powers of the integers which he relates to the figurate numbers. This leads him to the famous Bernoulli numbers, of huge importance in number theory, though in fact they had been already introduced by Johann Faulhaber in 1615–1631. The story is told in my *Pascal's Arithmetical Triangle*[3] where I was able to point out the hitherto unobserved pattern of the figurate numbers which lies behind the coefficients of the polynomials for the sums (thus enabling me to correct one of the coefficients in Bernoulli's table which had been reproduced repeatedly for 270 years without anyone noticing the error.)

Part III consists of 24 worked examples that explain, in Bernoulli's words, "the use of the preceding doctrine in various ways of casting lots and games of chance". I draw attention only to Problem XVII, which I happened to choose to work through only to find that I disagreed with Bernoulli's answer[4]. For those who would like to try their hand, the problem is as follows. In a version of roulette, the wheel is surrounded by 32 equal pockets marked 1 to 8 four times over. Four balls are released and are flung at random into the pockets, no more than one in each. The sum of the numbers of the four occupied pockets determines the prize (in francs, say) according to a table which Bernoulli gives. The cost of a throw is 4 francs. What is the player's expectation?

Of course, one needs the table to compute this – it is on the *Significance* website at `bit.ly/1bpbS1w` – but when I did so I came to a different answer than his, 4 + 153/17 980 = 4.0085 instead of 4 + 349/3596 = 4.0971. Which is correct?

So much for the first three parts. But it is the unfinished Part IV that makes this the foundation-stone of mathematical statistics.

Its title is "Civil, Moral and Economic Matters". It is only 30 pages long. In the first three of the five chapters Bernoulli completes the change from considering *expectation* as in games of chance to considering *probability* as a degree of certainty which can be estimated (as we should now say) from observing the outcomes of a sequence of events. This creation of the modern, mathematical definition of probability, and linking it to empirical observations in the physical world, is fundamental. But there is more. In Chapter IV Bernoulli introduces and explains his "golden theorem". Bernoulli himself recognises its importance, as witness his description in the box below. And, as he says, he seems to have been pondering it for twenty years before setting it down on paper.

His own description of it, again in the box below, is beautifully clear. Mathematically we can put it that the relative frequency of an event with probability $p = r/t$, $t = r + s$, in nt independent trials converges in probability to p with increasing n. Intuitively, it seems obvious: if we toss a fair coin a few times – say 10 – it might come up 5 heads and 5 tails, but it might well also come up 6/4, or 7/3. Toss it 100 times, and the ratio is much less likely to be very far from 50/50. Toss it 10 000 times and the ratio will be very close to 50/50 indeed. But intuition is a poor guide, especially in statistics and probability. For a sure foundation, we need proof – and Bernoulli gives us that proof. It follows in Chapter V. Thus, argues Bernoulli, we can infer with increasing certainty the unknown probability from a series of supposedly independent trials.

Ars conjectandi is the founding document of mathematical statistics because if his golden theorem were not true, mathematical statistics would be a house built on sand. It is not so built. The golden theorem confirms our intuition that the proportions of successes and failures in a stable sequence of trials really do converge on their postulated probabilities in a strict mathematical sense, and therefore may be used to estimate those probabilities. Mathematical statistics can therefore proceed.

Jakob Bernoulli's tombstone, Basel cathedral. Credit: Wladyslaw Sojka,

Bernoulli intended it to proceed. His plan was to extend it to all kinds of areas from economics to morality and law. How, for example, should a marriage contract divide the new family money fairly between the bride's father, the groom's father, and any children in the event of the bride or groom's death? It would depend, among other things, on the probabilities of the father dying before the son. That was a problem he had considered earlier.

But *Ars conjectandi* stops abruptly. The planned continuation to "economics and politics" is left for others to develop, with Bernoulli's golden theorem as their inspiration. Unfinished the book may be, but its influence had only just begun when it fell from the press of the Thurneysen Brothers in Basle three centuries ago.

Bernoulli's golden theorem – from *Ars conjectandi*[5]

"This is therefore the problem that I now want to publish here, having considered it closely for a period of twenty years, and it is a problem of which the novelty, as well as the high utility, together with its grave difficulty, exceed in weight and value all the remaining chapters of my doctrine…

"To illustrate this by an example, I suppose that without your knowledge there are concealed in an urn 3000 white pebbles and 2000 black pebbles, and in trying to determine the numbers of these pebbles you take out one pebble after another (each time replacing the pebble you have drawn before choosing the next, in order not to decrease the number of pebbles in the urn), and that you observe how often a white and how often a black pebble is withdrawn. The question is, can you do this so often that it becomes ten times, one hundred times, one thousand times, etc., more probable (that is, it be morally certain) that the numbers of whites and blacks chosen are in the same 3 : 2 ratio as the pebbles in the urn, rather than in any other different ratio?"

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In France, Denmark, Sweden, Belgium, lower Germany, and neighboring regions, a kind of game is played that they call Cinq et Neuf; it is played by two people, A and B, with two dice; one of them, A, receives unending[13] turns. These are the conditions of play: if A on the first turn throws a 3 or an 11 or any pair (*un doublet, ein Pasch*), that is, two ones, two twos, threes, etc., then A wins. If A throws a 5 or a 9, the other player, B, wins. If A throws any other number of points, namely 4, 6, 7, 8, or 10,[14] then neither of the players wins, but the game continues until a 5 or 9 is thrown, in which case B is always the winner, or until exactly the same number of points is thrown again as was thrown on the first throw, in which case A wins. The condition concerning the throwing of a 3 or 11 or of any pair does not aid A except on the first throw. On these assumptions what is the ratio of lots?

Since player A, to the extent that he may throw 4, 6, 7, 8, or 10 points on the first throw, arrives at lots so far unknown and unexplored, these must be investigated before all else.

Let us assume that A has thrown a 4 on his first turn and now is ready for another turn. Then, since with two dice there are three cases that produce the same 4 points and give A victory in the game and eight other cases that produce 5 or 9 points and argue for his loss of the stake, while all of the others oblige him to repeat the throw and therefore make no more difference than if they absolutely did not exist, by Corollary 4 to Proposition III of Part I for that reason his expectation becomes $[3(1) + 8(0)]/11 = 3/11$. Likewise, he acquires the same lot if the first throw is a 10, since with two dice 4 and 10 correspond to an equal number of cases. **[168]**

Next let us assume that the first throw is a 6. Then, since there are 5 cases in which, on another throw, the same 6 may reappear, while again there are 8 cases in which 5 or 9 points may occur, it follows that player A has 5 cases in his favor and 8 against him (neglecting, as before, the remaining cases, which leave him in the same status), which produces a lot for him of $[5(1) + 8(0)]/13 = 5/13$. His expectation is also just as great if on the first throw an 8 appears, since 6 and 8 are liable to the same number of cases.

Let us assume, finally, that the first throw is a 7. Then, since the same 7 can reappear on the following throw in 6 cases, there are now 6 cases for player A, while 8 as before are for the opponent. This makes the lot of A $[6(1) + 8(0)]/14 = 3/7$.

These results having been found, I proceed with the problem by considering the position of player A before the first throw and by examining in how many cases by this throw he may come to any of the preceding lots. First, it is clear that with two dice there are 6 cases of doubles which, together with the 4 additional cases producing 3 or 11 points, make 10 cases altogether that give A possession of the total stake according to the rule of the game. Then it is also apparent, as has already been said, that there are 8 cases producing 5 or 9 points by which A loses the whole stake. In addition, there are 6 cases of 4 and 10 points, but this includes 2 cases of pairs, of twos and fives, that give player A the whole stake and that have already been accounted for. Subtracting these there remain only 4 cases that bring him to the lot 3/11 found above. Next there are 10 cases of 6 or 8 points. Again subtracting 2 cases for the pairs of threes and fours, there are left 8 cases that advance him to the lot 5/13 found above. Finally, there are 6 remaining cases of 7 points in which, as we have shown above, he acquires a lot of 3/7. When all these are put together, it is clear that A's expectation at the beginning of the game will be $10(1) + 8(0) + 4(3/11) + 8(5/13) + 6(3/7)]/36 = 4189/9009$. Accordingly **[169]** B's is 4820/9009, so that the ratio of their lots is as 4189 to 4820. Whence it becomes clear that B's position is stronger than A's, most of all since there are those who judge otherwise and prefer to take A's part.

13. Latin: *perpetuas*. This means that the turns are in a row and also unlimited in number.
14. That is, when the even numbers are not produced by pairs.

Problem XVII

The valuation of the lot in a certain other kind of game of chance.

I remember once seeing here at the time of the weekly market a certain peddler[15] who was explaining the following kind of game in the marketplace and attracting to it those passing by. There was a circular disk made quite level, mounting upwards for a little while toward the center. The border was surrounded by 32 contiguous and equal small pockets or openings that were marked into four distinct classes or series by the numbers written four times in order from I up to VIII. A dice box hung perpendicularly over the middle of the disk. The one about to make a trial of fortune dropped through the cavity of the dice box four little balls to be received by an equal number of compartments in the circumference of the disk. He received as a prize what the sum of the numbers in the compartments indicated, which might be larger or smaller depending on the sum, as the prizes in the following chart indicate. For each throw of the balls, the player was to pay four coins. What is the player's expectation?

First, it is clear that any throw of the balls will produce at least 4 points and at most 32 points, each of which occurs in one case, 4 points only if each ball falls into the first of a series of compartments and 32 points if each ball falls into

15. Latin: *circulatorem*.

Points		*Coins*		*Cases*
4	32	120	180	1
5	31	100	32	16
6	30	30	25	52
7	29	24	24	128
8	28	18	16	245
9	27	10	12	416
10	26	6	8	664
11	25	6	6	976
12	24	6	4	1369
13	23	5	4	1776
14	22	3	3	2204
15	21	3	3	2560
16	20	3	3	2893
17	19	2	3	3088
18		2		3184

the last compartment of a series. Next, I observe that the number of cases is multiplied for the intermediate numbers of points, increasing as the distance from either extreme, 4 or 32, increases, and reaching the maximum number of cases when the designated number of points is 18, the arithmetic mean between 4 and 32, while pairs of numbers equally distant above and below 18 have equal numbers of cases. Third, I consider that, of the compartments into which the balls fall on any throw, either all four contain the same given number, or three have the same number and the **[170]** fourth a different one, or two are the same and the other two both contain the same different number, or two are the same and the other two have different numbers, or, finally, all four compartments contain different given numbers. Of these possibilities, the first can occur in a single case, the second in 16 cases, the third in 36, the fourth in 96, and the last in 256 cases. For instance, since there are four homologous compartments containing the same number, for example, I, if some of the balls, for instance three, are to be received by these positions, then it is clear that this can happen in as many cases as there are triples contained in 4 things, namely four. Further, if the fourth ball is to be received by a compartment with a different number, for instance II (which, because there are four units in 4 things can again occur in four ways), it may be concluded that four times four or 16 cases exist altogether that result in three balls being received by three compartments marked with the number I, while the fourth ball enters a compartment marked with the number II. In the same way

it is readily concluded, since there are six pairs of four things, that there are six times 6 or 36 cases in which it happens that two balls fall into compartments marked I and the other two balls fall into compartments marked II. Likewise, there are six times four times four, that is, 96, cases in which two balls fall into compartments marked I, the third ball falls into a compartment marked II, and the fourth ball into a compartment marked III, and there are $4 \times 4 \times 4 \times 4$, that is 256, cases in which one of the balls falls into a compartment marked I, the second into a compartment marked II, the third into a compartment marked III, and the fourth into a compartment marked IV. Here, finally, we should remark that we neglect the 24 variations that arise solely from the permutations of the 4 balls among themselves, or indeed we might have had just that many secondary cases produced by each of the primary cases. **[171]**

With these preliminaries understood, we are now ready to investigate the number of cases for any given number of points in the same way that, after Proposition IX of Part I, we inquired into the numbers of throws in dice. This is done, namely, by resolving the proposed number of points into 4 parts (because of the 4 balls) of which no part exceeds 8 (larger numbers not being ascribed to any of the compartments), and this in every way possible, and then ascribing to each of these ways the number of cases as determined above. The sum of these will be the number desired. And since, in exactly the same way as the number of cases was found for a given number of points, it is necessary for us to find the cases for all the points, we may follow another more compendious method and find all the cases in one operation in this way:

At the top of the following Table are written in the margin in order the numbers of points from IV up to XVIII. Note that it is sufficient to determine the cases for these numbers, since each of the numbers above XVIII has the same number of cases as some number below XVIII.

If we assume that all balls are received by 4 homologous compartments, then their sum is either four I's or four II's, III's, IV's, etc., of which the sums are 4, 8, 12, 16, etc. Therefore, put into the left margin 1.1.1.1 (understanding by this tacitly also 2.2.2.2 etc. up to 8.8.8.8) and in the same row under the numbers IV, VIII, XII, XVI, etc., put the number 1.

If we assume that three balls are received by homologous compartments and the fourth by a diverse compartment, the three like numbers are either three I's, or II's, III's, etc. If the like numbers are three I's, then the fourth number is either II, III, or IV, etc., which single throws produce the sums V, VI, VII, VIII, . . . XI. Therefore, put 1.1.1.2 in the margin (understanding by this also 1.1.1.3 etc. up to 1.1.1.8), and under V, VI, VII, . . . XI points write 16. If the three like numbers are II's, the fourth number may be a I, or a III or IV, etc., producing the sums VII, IX, X, . . . XIV. Therefore, put in the margin 2.2.2.1 (understanding

Points:

Combinations	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.
1.1.1.1	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.
1.1.1.2	.	16	16	16	16	16	16	16	.	.	.	.	.	.	.
2.2.2.1	.	.	.	16	.	16	16	16	16	16	16	.	.	.	.
3.3.3.1	.	.	.	.	.	.	16	16	.	16	16	16	16	16	.
4.4.4.1	.	.	.	.	.	.	.	.	.	16	16	16	.	16	16
5.5.5.1	.	.	.	.	.	.	.	.	.	.	.	.	16	16	16
1.1.2.2	.	.	36	.	36	.	36	.	36	.	36	.	36	.	36
2.2.3.3	.	.	.	.	.	.	36	.	36	.	36	.	36	.	36
3.3.4.4	.	.	.	.	.	.	.	.	.	.	36	.	36	.	36
4.4.5.5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	36
1.1.2.3	.	.	.	96	96	96	96	96	96	.	.	.	.	.	.
1.1.3.4	.	.	.	.	.	96	96	96	96	96	.	.	.	.	.
1.1.4.5	.	.	.	.	.	.	.	96	96	96	96	.	.	.	.
1.1.5.6	.	.	.	.	.	.	.	.	.	96	96	96	.	.	.
1.1.6.7	.	.	.	.	.	.	.	.	.	.	.	96	96	.	.
1.1.7.8	.	.	.	.	.	.	.	.	.	.	.	.	.	96	.
2.2.1.3	.	.	.	.	96	96	96	96	96	96	.	.	.	.	.
2.2.3.4	.	.	.	.	.	.	.	96	96	96	96	96	.	.	.
2.2.4.5	.	.	.	.	.	.	.	.	.	96	96	96	96	.	.
2.2.5.6	.	.	.	.	.	.	.	.	.	.	.	96	96	96	.
2.2.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
3.3.1.2	.	.	.	.	.	96	.	96	96	96	96	96	.	.	.
3.3.2.4	.	.	.	.	.	.	.	.	96	96	96	96	96	.	.
3.3.4.5	.	.	.	.	.	.	.	.	.	.	.	96	96	96	96
3.3.5.6	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
4.4.1.2	.	.	.	.	.	.	.	96	96	.	96	96	96	96	.
4.4.2.3	.	.	.	.	.	.	.	.	.	96	.	96	96	96	96
4.4.3.5	.	.	.	.	.	.	.	.	.	.	.	.	96	96	96
5.5.1.2	.	.	.	.	.	.	.	.	.	96	96	96	.	96	96
5.5.2.3	.	.	.	.	.	.	.	.	.	.	.	96	96		96
5.5.3.4	.	.	.	.	.	.	.	.	.	.	.	.	.	96	.
6.6.1.2	.	.	.	.	.	.	.	.	.	.	.	96	96	96	96
6.6.2.3	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
7.7.1.2	.	.	.	.	.	.	.	.	.	.	.	.	.	96	96
1.2.3.4	.	.	.	.	.	.	256	256	256	256	256	.	.	.	.
1.2.4.5	.	.	.	.	.	.	.	.	256	256	256	256	.	.	.
1.2.5.6	.	.	.	.	.	.	.	.	.	.	256	256	256	.	.
1.2.6.7	.	.	.	.	.	.	.	.	.	.	.	.	256	256	.
1.2.7.8	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
1.3.4.5	.	.	.	.	.	.	.	.	.	256	256	256	256	.	.
1.3.5.6	.	.	.	.	.	.	.	.	.	.	.	256	256	256	.
1.3.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	256	256
1.4.5.6	.	.	.	.	.	.	.	.	.	.	.	.	256	256	256
1.4.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
2.3.4.5	.	.	.	.	.	.	.	.	.	.	256	256	256	256	.
2.3.5.6	.	.	.	.	.	.	.	.	.	.	.	.	256	256	256
2.3.6.7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
2.4.5.6	.	.	.	.	.	.	.	.	.	.	.	.	.	256	256
3.4.5.6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	256
Sum of cases	1	16	52	128	245	416	664	976	1369	1776	2204	2560	2893	3088	3184

Total of all the cases = 35,960.

by this also 2.2.2.3 etc.) and in the space under the numbers VII, IX, X, . . . XIV again write 16. One should proceed similarly where the homologous numbers are three III's with the fourth a I, II, IV, or V, etc., or **[172]** three IV's, with the fourth a I, II, III, or V, etc., or three V's, etc., where the fourth is always one of the remaining numbers, writing 16 under each of the sums of points that are produced by the addition of the 4 numbers in the compartments.

If we assume, next, that of the compartments two have the same number and the remaining two similarly contain the same number, but not the number in the first two compartments, then the numbers in the compartments are either two I's with two II's, III's, IV's, etc., which added to the I's make sums of VI, VIII, X, . . . XVIII, or they are two II's with two III's, IV's, etc., which added to the II's make sums of X, XII, XIV, etc. Or they may be two III's with just as many IV's, etc., or two IV's with two V's, etc., and so forth. Therefore, put in the margin of the Table 1.1.2.2, 2.2.3.3., 3.3.4.4, etc. (omitting the others, e.g., 1.1.3.3, 1.1.4.4, etc., not to mention 2.2.4.4, etc., 3.3.5.5, for the sake of economy) and under the sums of these numbers (and also of the sums of the others understood as similar) write 36.

Then we continue by assuming that two balls are in like compartments, while the remaining two are in compartments with numbers different from these and from each other. Again the two like numbers can be two I's, or two II's, or two III's, etc., and, if they are I's, then the third may be a II with the fourth a III, IV, V, etc., or the third may be a III with the fourth a IV, V, etc., and so forth. If the two like numbers are II's, then the third can be a I with the fourth a III, IV, V, VI, etc., and so forth, or the third can be a III with the fourth a IV, V, VI, etc., or it can be a IV, with the fourth a V, VI, etc. etc. And if the two like numbers are two III's, the third may be a I with the fourth a II, IV, V, VI, etc., or a II with the fourth a IV, V, VI, etc., or IV with the fourth a 5, 6, etc. etc. If the two like numbers are IV's, the third can be either I with the fourth II, III, V, etc., or a II with the fourth a III, V, etc. etc., and so likewise for all the rest. Because this is so, write the first of these combinations 1.1.2.3, 1.1.3.4, etc., and 2.2.1.3 etc. 3.3.1.2 etc. in the margin, understanding by these also the other similar combinations, and under the sums so produced by each group of four numbers, write 96.

Finally, if we assume that the compartments into which the 4 balls fall all have different numbers, there are these combinations: 1, 2, 3, with the fourth 4, 5, 6, etc.; 1, 2, 4, with the fourth 5, 6, 7, etc. etc. **[173]** Also 1, 3, 4; 1, 3, 5; etc., 1, 4, 5, etc. with the fourth etc., not to mention 2, 3, 4; 2, 3, 5, etc., with the fourth a different number etc. etc. etc. up to 3, 4, 5, 6, with which all the possible combinations are complete. Therefore, with the first of these combinations written in the margins and the rest omitted, write 256 under each of the sums of 4 numbers just as can be observed displayed in the adjoining Table.

Therefore, adding in one sum the numbers of cases in each column, the result is the total number of cases for each number of points, namely 1 case in which IV points are obtained, 16 cases in which V points are obtained, 52 for VI, and so forth up to XVIII points, which may be obtained twice 16, 4 times 36, ten times 96, and eight times 256, that is, altogether, in 3184 cases. Since the numbers of points over XVIII each agree in number of cases one-to-one with points below XVIII, for example XIX with XVII, XX with XVI, etc., as we pointed out at the beginning and as can easily be shown, it follows, if the numbers of cases for the points from IV to XVII are doubled, and if the result, 32,776, is added to the 3184 cases in which XVIII results, that the sum 35,960 is the total of all the cases whatsoever. That this enumeration is correctly done and that no combination has been omitted can be shown by the fact that the number of groups of four among 32 things (that is, compartments) is found to be exactly equal, namely (by Chapter IV of Part II) $(32 \cdot 31 \cdot 30 \cdot 29)/(1 \cdot 2 \cdot 3 \cdot 4) = 35{,}960$.

Once the numbers of cases are found for each number of points, the remaining steps are exceedingly easy and can be accomplished by Proposition III of Part I. The numbers of cases are multiplied by the prizes that are obtained in each case. So, since IV points obtains a prize of 120 coins as shown in the chart above XXXII points obtains 180, V points obtains 100 coins, XXXI points obtains 32 coins, VI points 30, XXX points 25, etc., multiply 1 case by 120 and again by 180; 16 cases by 100 and also by 32; 52 cases by 30 and by 25; etc., or, more briefly, 1 by 300 = 120 + 180, 16 by 132 = 100 + 32, 52 by 55 = 30 + 25, and so forth up to 3184 multiplied by 2, and finally divide the sum of all **[174]** these products by 35,960, the sum of all the cases. The quotient, 4 349/3596, is the expectation of the player. Since, by hypothesis, the cost of each throw is only 4 coins, it is apparent that the player's lot is greater than that of the peddler, and accordingly that by this sort of game, unless the prizes are decreased, the peddler cannot profit much.

Problem XVIII

On the card game called in the vernacular Trijaques.[16]

Very common among the Germans is the kind of game called Trijaques, which has an affinity to the French game Brelan. From a deck of cards 24 are taken (with the rest set aside), six from each suit, the nines, tens, jacks, queens, kings, and aces, which hereafter will be referred to by their initials N.T.J.Q.K.A. These cards have the following priorities: in first place is the ace, followed by the king,

16. In English "Three Jacks."